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Scattering and Inverse Scattering for Steplike Potentials in the Schrödinger Equation

AMY COHEN & THOMAS KAPPELER

Introduction and Summary

This paper studies both the forward and inverse problems in the scattering theory of the Schrödinger equation

$$(0.1) \qquad -y'' + v(x)y = k^2 y, \qquad x \in \mathbf{R}$$

with "steplike" potentials $v(x)$ which are asymptotic to different constants as $x \to +\infty, -\infty$. The forward problem is to define appropriate scattering data for potentials in a reasonable class of potentials and to study their properties; the inverse problem is twofold: to give sufficient conditions on candidate scattering data to assure that these data are indeed the scattering data of some potential and to give a method for constructing that potential from that scattering data.

Analogous problems on the half-line were studied by Agranovich and Marchenko [1]. This problem for potentials decaying to 0 at both $+\infty$ and $-\infty$ has been studied extensively by Faddeev [7] and by Deift and Trubowitz [6]. There are important applications of these constructions to the solution and analysis of the Korteweg-deVries equation [13], [3], [8].

The problem for steplike potentials on the full line was first studied by Buslaev and Fomin [2]. Following the methods of Faddeev [7], Buslaev and Fomin developed a set of conditions which they claimed were necessary and sufficient that candidate scattering data arise from a potential v such that

$$(0.2) \qquad \int_{-\infty}^{0} |v(x)|(1 + |x|)dx < \infty; \qquad \int_{0}^{+\infty} |v(x) - c^2|(1 + |x|)dx < \infty.$$

A brief summary of the Buslaev-Fomin paper appears in the English translation [11] of a paper by Hruslov, who applies it to the Korteweg-deVries equation.

Unfortunately the paper of Buslaev and Fomin is incorrect: there is a counterexample [4] to their theorem, which is analogous to the counterexample to Faddeev's theorem in [7], found by Deift and Trubowitz [6]. Deift and Trubowitz formulated a revised version of Faddeev's theorem which could be proved cor-

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rectly by Faddeev's method; they also provided a new construction of the potential from the scattering data.

In this paper we both correct and extend the analysis of Buslaev-Fomin. For $c \geq 0$ and $N \geq 1$, let $P(c,N)$ denote the class of potentials $v(x)$ such that

$$v(x) \text{ is real valued on } \mathbf{R};$$

$$\int_{-\infty}^{\infty} |v(x) - c^2 H(x)|(1 + |x|^N)dx < \infty; \tag{0.3}$$

where $H(x)$ is the Heavyside function, i.e. $H(x) = 1$ if $x > 0$, $H(x) = 0$ if $x < 0$. Say that a potential $v(x)$ is of "generic" type if the Jost functions for (0.1) at $k = 0$ are independent; say that $v(x)$ is "exceptional" otherwise. (The Jost functions for (0.1) at $k = 0$ are the solutions $f_+(x,0)$ and $f_-(x,0)$ of $-y'' + v(x)y = 0$ such that $f_+(x,0)$ is asymptotic to e^{-cx} as $x \to +\infty$ and $f_-(x,0)$ is asymptotic to 1 as $x \to -\infty$.)

Our main result gives necessary and sufficient conditions for the scattering data of potentials in $P(c,N)$ with $N \geq 2$. For $P(c,1)$ with $c > 0$ we give sufficient conditions which are slightly stronger than our necessary conditions.

For $P(c,N)$ with $N \geq 2$, our primary interest is in the case $c > 0$. But the methods apply also when $c = 0$, and for $c = 0$ our results reduce to those of Deift and Trubowitz. For $c > 0$, Buslaev and Fomin considered only the generic case. Our results not only provide a correct treatment of the generic case, but also treat the exceptional case.

As far as we know there is not yet a published solution of the very interesting problem of complete characterization of the scattering data for potentials in $P(c,1)$ for either $c = 0$ or for $c > 0$.

We use (*) to denote complex conjugates. There is one other definition which should be reviewed before starting to work. We use H^{2+} for the Hardy space of functions f such that

$$f(k) \text{ is analytic in } \operatorname{Im} k > 0$$

and

$$\sup_{\eta>0} \int_{-\infty}^{\infty} |f(\xi + i\eta)|^2 d\xi < \infty.$$

For $f \in H^{2+}$, the sections $\phi_\eta(\xi) \equiv f(\xi + i\eta)$ for $\eta > 0$ converge in $L^2(\mathbf{R})$ as $\eta \downarrow 0$ to a limit denoted simply $f(\xi)$. Further, if we take the transform

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(\xi)e^{-2ix\xi}d\xi,$$

then $\hat{f}(x) = 0$ for all $x < 0$. Conversely if f is defined by

$$f(k) = \int_0^{\infty} A(x)e^{+2ixk}dx$$

for some $A(x)$ in $L^2(\mathbf{R}^+)$, then $f \in H^{2+}$.

Chapter I, in three sections, presents the forward scattering theory. This follows Faddeev [7] and Buslaev-Fomin [2], but is more detailed where necessary. Some results are also taken from Deift and Trubowitz [6]. Chapter II, also in three sections, presents the inverse problem. The approach is basically that of Faddeev, but care is required to avoid Faddeev's errors and the pitfalls arising from the fact that $c > 0$ and therefore that $\sqrt{z^2 + c^2}$ cannot be analytic in Im $z \geq 0$.

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Chapter I: The Forward Scattering Theory

Section 1. Definition of scattering data. We consider a potential $v(x)$ in the Schrödinger equation

$$-y'' + v(x)y = k^2 y \tag{1.1}$$

subject to the hypothesis that $v \in P(c,1)$:

$$v \text{ is real-valued on } \mathbf{R}, \tag{1.2a}$$

$$\int_{-\infty}^{\infty} |v(x) - c^2 H(x)|(1 + |x|)dx < \infty, \tag{1.2b}$$

where $H(x)$ is the Heavyside function.

Note that (1.1) is equivalent to

$$-y'' + \{v(x) - c^2\}y = \ell^2 y \tag{1.3}$$

where $\ell^2 = k^2 - c^2$. More precisely we define $\ell = \ell(k)$ for Im $k \geq 0$ by choosing the branch of $\ell = \sqrt{k^2 - c^2}$ with Im $\ell \geq 0$. Thus $\ell(k)$ is analytic for Im $k > 0$ and continuous for Im $k \geq 0$. The map $k \to \ell(k)$ is one-to-one from $\{k : \text{Im } k > 0\}$ onto $\{\ell : \text{Im } \ell > 0\} \sim \{i\lambda : 0 < \lambda \leq c\}$. We shall write $k = k(\ell)$ for the inverse function, and note that $k(\ell)$ is analytic on $\{\ell : \text{Im } \ell > 0\} \sim \{i\lambda : 0 < \lambda \leq c\}$ but has a jump across the slit, $\{i\lambda : 0 < \lambda \leq c\}$.

The Jost functions, $f_\pm(x,k)$, are the solutions of (1.1) such that

$$e^{-i\ell x} f_+(x,k) \to 1 \quad \text{and} \quad e^{-i\ell x} f'_+(x,k) \to i\ell \qquad \text{as } x \to +\infty, \tag{1.4}$$

$$e^{ikx} f_-(x,k) \to 1 \quad \text{and} \quad e^{ikx} f'_-(x,k) \to -ik \qquad \text{as } x \to -\infty. \tag{1.5}$$

The hypotheses (1.2) are sufficient to assure the existence of $f_\pm(x,k)$ for Im $k \geq 0$. A detailed construction can be found in [6, §2]; here we merely outline the analysis.

Since $\int_{-\infty}^{X} |v(x)|(1 + |x|)dx < \infty$ for all finite X, the construction of f_- follows [6] exactly. One gets

$$f_-(x,k) = e^{-ikx}\left\{1 + \int_{-\infty}^{0} B_-(x,y)e^{-2iky}dy\right\} \tag{1.6}$$

where B_- is determined by the integral equation

$$B_-(x,y) = \int_{z=-\infty}^{x+y} v(z)dz - \int_{z=0}^{y}\int_{s=-\infty}^{x+y-z} v(s)B_-(s,z)ds\,dz \tag{1.7}$$

for $x \in \mathbf{R}$, $y \leq 0$. From (1.2) and (1.7) it follows that

$B_-(x,y)$ is continuous in $\mathbf{R} \times (-\infty,0]$;

$\partial_x B_-$ and $\partial_y B_-$ exist almost everywhere;

$\int_{-\infty}^{0} |\partial_x^\alpha \partial_y^\beta B_-(x,y)|(1 + |y|)dy < \infty$ for all x if $\alpha + \beta \in \{0,1\}$;

$v(x) = \partial_x B_-(x,0)$ almost everywhere;

$f_-(x,k)$ is analytic for $\operatorname{Im} k > 0$ and continuous for $\operatorname{Im} k \geq 0$.

To discuss $f_+(x,k)$ it is better to start with the equivalent form (1.3) of the Schrödinger equation since

$$\int_{X}^{\infty} |v(x) - c^2|(1 + |x|)dx < \infty$$

for all finite X. For $\operatorname{Im} \ell \geq 0$, the Deift and Trubowitz construction yields

$$\tilde{f}_+(x,\ell) = e^{i\ell x}\left\{1 + \int_{0}^{\infty} B_+(x,y)e^{2i\ell y}dy\right\} \tag{1.8}$$

where

$$B_+(x,y) = \int_{z=x+y}^{\infty} \{v(z) - c^2\}dz + \int_{z=0}^{y}\int_{s=x+y-z}^{\infty} \{v(s) - c^2\}B_+(s,z)dsdz \tag{1.9}$$

for $x \in \mathbf{R}$, $y \geq 0$. Analysis of (1.9) shows that

$B_+(x,y)$ is continuous in $\mathbf{R} \times [0,\infty)$;

$\partial_x B_+$ and $\partial_y B_+$ exist almost everywhere;

$\int_{0}^{\infty} |\partial_x^\alpha \partial_y^\beta B_+(x,y)|(1 + |y|)dy < \infty$ for all x if $\alpha + \beta \in \{0,1\}$;

$v(x) - c^2 = -\partial_x B_+(x,0)$ almost everywhere;

and finally

$$\tilde{f}_+(x,\ell) \text{ is analytic for } \operatorname{Im} \ell > 0 \text{ and continuous on } \operatorname{Im} \ell \geq 0.$$

To write the Jost function as a function of k we set

$$f_+(x,k) = \tilde{f}_+(x,\ell(k)).$$

By the definition of $\ell(k)$, $f_+(x,k)$ is then analytic in $\operatorname{Im} k > 0$ and continuous on $\operatorname{Im} k \geq 0$.

It is convenient to extend the domains of B_+ and B_- so that

$$B_+(x,y) = 0 \text{ if } y < 0; \qquad B_-(x,y) = 0 \text{ if } y > 0.$$

For each real k, the four functions f_+, f_+^*, f_-, and f_-^* satisfy the same real ordinary differential equation. Computing Wronskians we find that

$$W[f_-,f_-^*] = \det\begin{bmatrix} e^{-ikx} & e^{+ikx} \\ -ike^{-ikx} & ike^{ikx} \end{bmatrix} = 2ik \qquad \text{for all } k \in \mathbf{R}$$

$$W[f_+^*,f_+] = \det\begin{bmatrix} e^{-i\ell x} & e^{i\ell x} \\ -i\ell e^{-i\ell x} & i\ell e^{i\ell x} \end{bmatrix} = 2i\ell \qquad \text{for all } k \in \mathbf{R} \text{ with } |k| \geq c.$$

If $k \in \mathbf{R}$ and $|k| < c$, then $\ell(k)$ is pure imaginary, $f_+(x,k)$ is real valued, and $W[f_+^*,f_+] = 0$. Thus we see that f_- and f_-^* are independent solutions of (1.1) for $k \in \mathbf{R}$ and $k \neq 0$, whereas f_+ and f_+^* are independent solutions for $k \in \mathbf{R}$ and $|k| > c$. We may now define the coefficients $a_\pm(k)$ and $b_\pm(k)$ by the relations

$$f_+(x,k) = a_-(k)f_-^*(x,k) + b_-(k)f_-(x,k) \qquad \text{for } 0 \neq k \in \mathbf{R} \tag{1.10}$$

and

$$f_-(x,k) = a_+(k)f_+^*(x,k) + b_+(k)f_+(x,k) \qquad \text{for } k \in \mathbf{R},\ |k| > c. \tag{1.11}$$

Taking Wronskians one finds that if $k \in \mathbf{R}$ and $|k| > c$, then

$$\begin{aligned} 2i\ell a_+(k) &= W[f_-(x,k),f_+(x,k)] \\ 2i\ell b_+(k) &= W[f_+^*(x,k),f_-(x,k)]. \end{aligned} \tag{1.12}$$

Similarly, if $k \in \mathbf{R}$ and $k \neq 0$, then

$$\begin{aligned} 2ika_-(k) &= W[f_-(x,k),f_+(x,k)] \\ 2ikb_-(k) &= W[f_+(x,k),f_-^*(x,k)]. \end{aligned} \tag{1.13}$$

We use equations (1.12) and (1.13) to extend the domains of a_- and b_- to $\mathbf{R} \sim \{0\}$ and of a_+ and b_+ to $\mathbf{R} \sim \{+c,-c\}$.

Lemma 1.1.

(i) $a_-(-k) = a_-^*(k)$ *and* $b_-(-k) = b_-^*(k)$ *for* $0 \neq k \in \mathbf{R}$.

(ii) $a_+(-k) = a_+^*(k)$ *and* $b_+(-k) = b_+^*(k)$ *for* $k \in \mathbf{R} \sim \{c,-c\}$.

(iii) $\ell a_+(k) = ka_-(k)$ *and* $\ell b_+(k) = -kb_-^*(k)$ *for* $k \in \mathbf{R} \sim \{-c,0,c\}$.

(iv) $b_+(k) = -a_+(k)$ *for* $-c < k < c$; $b_-(k) = a_-^*(k)$ *for* $k \in \mathbf{R}$, $0 < |k| \le c$.
(v) $\ell|a_+|^2/k = k|a_-|^2/\ell = 1 + k|b_-|^2/\ell = 1 + \ell|b_+|^2/k$ *and*
$0 = a_+b_- + a_-b_+^* = a_+^*b_- + a_-b_+$ for $k \in \mathbf{R}$, $|k| > c$.

Proof. (i) through (iv) follow from (1.12) and (1.13). Part (v) is found by substituting (1.10) into (1.11) and vice versa.

Lemma 1.2. *Let $W(k)$ denote the Wronskian $W[f_-(x,k), f_+(x,k)]$. Then $W(k)$ is continuous for* Im $k \ge 0$, *and analytic for* Im $k > 0$. *Further $W(k) \neq 0$ for $0 \neq k \in \mathbf{R}$.*

Proof. We have

$$W(k) = f_-(x,k)f_+'(x,k) - f_+(x,k)f_-'(x,k)$$

and we already know that f_+ and f_- are continuous for Im $k \ge 0$ and analytic for Im $k > 0$. Because $\partial_x B_+(x,y)$ and $\partial_x B_-(x,y)$ are integrable functions of y, we get

$$f_-'(x,k) = -ikf_-(x,k) + e^{-ikx}\int_{-\infty}^{0} \partial_x B_-(x,y)e^{-2iky}dy$$

and

$$f_+'(x,k) = i\ell f_+(x,k) + e^{i\ell x}\int_{0}^{\infty} \partial_x B_+(x,y)e^{2i\ell y}dy,$$

whence f_+' and f_-' are continuous in Im $k \ge 0$ and analytic in Im $k > 0$. Thus W is continuous in Im $k \ge 0$ and analytic in Im $k > 0$.

Finally, we look at $W(k)$ for real k. If $|k| > c$, then $a_-(k) \neq 0$ by part (v) of Lemma 1.1, and $W(k) = 2ika_-(k) \neq 0$ by (1.13). Suppose there were a real k_0 such that $0 < |k_0| \le c$ and $W(k_0) = 0$. Then both $a_-(k_0) = 0$ by (1.14) and $b_-(k_0) = 0$ by part (iv) of Lemma 1.1. But then $f_+(x,k_0) = 0$ for all x by (1.10), which contradicts the defining condition (1.13) on f_+. Thus $W(k)$ cannot vanish for any nonzero real k. □

At many points in this paper we will need different arguments depending on whether or not $W(0) = 0$. The term "generic," defined in the introduction, describes the case $W(0) \neq 0$. The term "exceptional" describes the case $W(0) = 0$.

As a consequence of Lemma 1.2 we can use (1.12) and (1.13) to extend the domains of a_+ and a_- so they are analytic in Im $k > 0$.

Lemma 1.3. *Suppose $a_-(k_0) = 0$ and* Im $k_0 > 0$. *Then k_0 is pure imaginary and*

$$\dot{a}_-(k_0) = -i\int_{-\infty}^{\infty} f_+(x,k_0)f_-(x,k_0)dx \neq 0.$$

Proof. By (1.13), $f_+(x,k_0)$ and $f_-(x,k_0)$ are dependent solutions of (1.1); thus

$$f_-(x,k_0) = \mu_0 f_+(x,k_0), \qquad \mu_0 \neq 0.$$

The asymptotic conditions (1.4), (1.5) tell us that $f_+(x,k_0)$ decays exponentially as $x \to +\infty$ and $f_-(x,k_0)$ decays exponentially as $x \to -\infty$. Since $f_- = \mu_0 f_+$ both $f_+(x,k_0)$ and $f_-(x,k_0)$ decay exponentially at both $+\infty$ and $-\infty$. So $f_\pm(x,k_0)$ are $L^2(\mathbf{R})$ eigenfunctions of the operator $-D^2 + v$, with eigenvalue $(k_0)^2$. Thus $(k_0)^2 \in \mathbf{R}$. But since $\operatorname{Im} k_0 > 0$, it must be that k_0 is pure imaginary. We shall write $k_0 = i\kappa_0$, $\kappa_0 > 0$.

Next we compute $\dot{a}_-(k_0)$, introducing the convention that dot ($\dot{}$) denotes the k-derivative, while reserving prime ($'$) for the x-derivative. By (1.13)

$$2ia_-(k) + 2ik\dot{a}_-(k) = \dot{W}(k) = W[\dot{f}_-,f_+] + W[f_-,\dot{f}_+].$$

Setting $k = k_0$ we get

$$2ik_0\dot{a}_-(k_0) = W[\dot{f}_-,f_+]|_{k_0} + W[f_-,\dot{f}_+]|_{k_0}. \tag{1.14}$$

Although $W(k)$ and $\dot{W}(k)$ are independent of x, the two Wronskians in (1.14) are not. For $\operatorname{Im} k > 0$, let

$$W_1(x,k) = W[\dot{f}_-,f_+] = \dot{f}_-f'_+ - f_+\dot{f}'_-$$

$$W_2(x,k) = W[f_-,\dot{f}_+] = f_-\dot{f}'_+ - \dot{f}_+f'_-.$$

From (1.1) we have

$$f''_\pm = (v(x) - k^2)f_\pm,$$

whence

$$\dot{f}''_\pm = (v(x) - k^2)\dot{f}_\pm - 2kf_\pm.$$

Then

$$\begin{aligned} \partial_x W_1(x,k) &= [\dot{f}_-f'_+ - f_+\dot{f}'_-]' \\ &= \dot{f}'_-f'_+ + \dot{f}_-f''_+ - f'_+\dot{f}'_- - f_+\dot{f}''_- \\ &= \dot{f}_-[(v - k^2)f_+] - f_+[(v - k^2)\dot{f}_- - 2kf_-] \\ &= +2kf_+f_-. \end{aligned} \tag{1.15}$$

Since $0 = \partial_x W = \partial_x[W_1 + W_2]$, we get

$$\partial_x W_2(x,k) = -\partial_x W_1(x,k) = -2kf_+f_-. \tag{1.16}$$

Setting $k = k_0$ in (1.15) and (1.16) we get

$$\partial_x W_1(x,k_0) = 2k_0 f_+(x,k_0)f_-(x,k_0) \tag{1.17}$$

$$\partial_x W_2(x,k_0) = -2k_0 f_+(x,k_0)f_-(x,k_0). \tag{1.18}$$

The trick now is to integrate (1.17) and (1.18) over half-lines chosen so that we can be sure the limits at infinity will vanish. Integrate (1.17) from $-\infty$ to 0 to get

$$\int_{-\infty}^0 \partial_x W_1(x,k_0)dx = 2k_0 \int_{-\infty}^0 f_+(x,k_0)f_-(x,k_0)dx,$$

or

$$W_1(0,k_0) - \lim_{x\to-\infty} W_1(x,k_0) = 2k_0 \int_{-\infty}^{0} f_+(x,k_0)f_-(x,k_0)dx. \tag{1.19}$$

Multiply (1.18) by -1 and then integrate from 0 to $+\infty$ getting

$$-\int_0^\infty \partial_x W_2(x,k_0)dx = 2k_0 \int_0^\infty f_+(x,k_0)f_-(x,k_0)dx$$

or

$$W_2(0,k_0) - \lim_{x\to+\infty} W_2(x,k_0) = 2k_0 \int_0^\infty f_+(x,k_0)f_-(x,k_0)dx. \tag{1.20}$$

Once we verify that these limits do indeed vanish, we can use (1.14) to argue that

$$\begin{aligned} 2ik_0\dot{a}_-(k_0) &= W_1(x,k_0) + W_2(x,k_0) \qquad \text{for all } x \\ &= W_1(0,k_0) + W_2(0,k_0) \\ &= 2k_0 \int_{-\infty}^\infty f_+(x,k_0)f_-(x,k_0)dx. \end{aligned}$$

Since $k_0 = i\kappa_0$ with $\kappa_0 > 0$, it follows that $k_0 \neq 0$ and

$$\dot{a}_-(k_0) = -i \int_{-\infty}^\infty f_+(x,k_0)f_-(x,k_0)dx.$$

Further we know that $f_-(x,k_0) \sim e^{\kappa_0 x}$ as $x \to -\infty$, so $f_-(x,k_0)$ is real valued and not identically zero. Thus

$$\dot{a}_-(k_0) = -i\mu_0^{-1} \int_{-\infty}^\infty [f_-(x,k_0)]^2 dx \neq 0.$$

It remains to verify that the limits at infinity in (1.19) and (1.20) do really vanish. Consider first

$$W_1(x,k_0) = f_-^{\cdot}(x,k_0)f_+'(x,k_0) - f_+(x,k_0)f_-^{\prime\cdot}(x,k_0).$$

Since

$$f_-(x,k_0) = \mu_0 f_+(x,k_0) \qquad \text{for all } x$$

we also have

$$f_-'(x,k_0) = \mu_0 f_+'(x,k_0) \qquad \text{for all } x.$$

Thus

$$W_1(x,k_0) = \mu_0^{-1}\{f_-^{\cdot}(x,k)f_-'(x,k_0) - f_-(x,k_0)f_-^{\prime\cdot}(x,k_0)\}.$$

Now

$$f_-(x,k) = e^{-ikx}\left\{1 + \int_{-\infty}^{0} B_-(x,y)e^{-2iky}dy\right\}$$

$$f'_-(x,k) = -ikf_-(x,k) + e^{-ikx}\int_{-\infty}^{0} \partial_x B_-(x,y)e^{-2iky}dy$$

$$\dot{f}_-(x,k) = -\, ixf_-(x,k) - 2ie^{-ikx}\int_{-\infty}^{0} B_-(x,y)\; ye^{-2iky}dy$$

$$\dot{f}'_-(x,k) = -if_-(x,k) - ixf'_-(x,k) - 2ke^{-ikx}\int_{-\infty}^{0} B_-(x,y)ye^{-2iky}dy$$

$$-2ie^{-ikx}\int_{-\infty}^{0} \partial_x B_-(x,y)ye^{-2iky}dy.$$

It can be easily verified that $\|B_-(x,\cdot)\| \leq M(x)$ and $\|\partial_x B_-(x,\cdot)\| \leq M(x)$ where $M(x)$ is some nondecreasing function and the norms are L^1 norms. Using the facts that $\kappa_0 > 0$ and that $ye^{2\kappa_0 y}$ is bounded for $-\infty < y < 0$, one can then show that all four quantities $|f_-(x,k_0)|$, $|f'_-(x,k_0)|$, $|\dot{f}_-(x,k_0)|$, and $|\dot{f}'_-(x,k_0)|$ approach 0 as $x \to -\infty$. Therefore

$$\lim_{x\to-\infty} W_1(x,k_0) = 0.$$

Similarly one can express $W_2(x,k_0)$ as

$$W_2(x,k_0) = f_-(x,k_0)\dot{f}'_+(x,k_0) - \dot{f}_+(x,k_0)f'_-(x,k_0)$$

$$= \mu_0 f_+(x,k_0)\dot{f}'_+(x,k_0) - \mu_0\dot{f}_+(x,k_0)f'_+(x,k_0)$$

and verify that $f_+(x,k_0)$, $\dot{f}'_+(x,k_0)$, $\dot{f}_+(x,k_0)$, and $f'_+(x,k_0)$ each have limit 0 as $x \to +\infty$. Thus

$$\lim_{x\to+\infty} W_2(x,k_0) = 0.$$

This completes the proof of Lemma 1.3. □

We shall call a function f "piecewise absolutely continuous" if there is a finite partition of $\mathbf{R}$, $-\infty = x_0 < x_1 < \ldots < x_N < x_{N+1} = +\infty$, such that f is absolutely continuous on each (x_n,x_{n+1}) where $0 \leq n \leq N$. Thus f is allowed only finite jump discontinuities at the x_n, f is differentiable almost everywhere, and f' is locally integrable.

Lemma 1.4.

(i) $a_-(k) = 1 + O(k^{-1})$ *as* $|k| \to \infty$ *with* $\operatorname{Im} k \geq 0$.

(ii) $b_-(k) = O(k^{-1})$ *as* $k \to \pm\infty$, $k \in \mathbf{R}$.

(iii) Suppose v *is piecewise absolutely continous,* $v \in P(c,1)$, $v' \in L^1(\mathbf{R})$. *Then* $b_-(k) = O(k^{-2})$ *as* $k \to \pm\infty$, $k \in \mathbf{R}$.

(iv) Suppose v is absolutely continuous, v' is piecewise absolutely continuous and L^1, and $v'' \in L^1(\mathbf{R})$. Then $b_-(k) = O(k^{-3})$ as $k \to \pm\infty$, $k \in \mathbf{R}$.

Proof (i). Let

$$h_-(x,k) = 1 + \int_{-\infty}^{0} B_-(x,y)e^{-2iky}dy, \tag{1.21}$$

$$h_+(x,\ell) = 1 + \int_{0}^{\infty} B_+(x,y)e^{2i\ell y}dy. \tag{1.22}$$

For all k in $\mathbf{R} \sim \{0\}$ and all x in $\mathbf{R}$, we have

$$2ika_-(k) = W[f_-,f_+] = W[e^{-ikx}h_-,e^{i\ell x}h_+].$$

Setting $x = 0$ in the Wronskian

$$\begin{aligned} 2ika_-(k) &= \det\begin{bmatrix} h_- & h_+ \\ -ikh_- + h'_- & i\ell h_+ + h'_+ \end{bmatrix} \\ &= i(k+\ell)h_-(0,k)h_+(0,\ell) + h_-(0,k)h'_+(0,\ell) - h'_-(0,k)h_+(0,\ell). \end{aligned} \tag{1.23}$$

Under (1.2) we know that $B_+^{(\alpha,\beta)}(x,\cdot) \in L^1(\mathbf{R}^+)$ and $B_-^{(\alpha,\beta)}(x,\cdot) \in L^1(\mathbf{R}^-)$ for $\alpha + \beta \leq 1$. Thus we can integrate by parts to get

$$h_-(0,k) - 1 = \frac{-B_-(0,0)}{2ik} + \frac{1}{2ik}\int_{-\infty}^{0} B_-^{(0,1)}(0,y)e^{-2iky}dy \tag{1.24}$$

and

$$h_+(0,\ell) - 1 = \frac{-B_+(0,0)}{2i\ell} - \frac{1}{2i\ell}\int_{0}^{\infty} B_+^{(0,1)}(0,y)e^{2i\ell y}dy. \tag{1.25}$$

One also gets

$$h'_-(0,k) = \int_{-\infty}^{0} B_-^{(1,0)}(0,y)e^{-2iky}dy \tag{1.26}$$

$$h'_+(0,\ell) = \int_{0}^{\infty} B_+^{(1,0)}(0,y)e^{2i\ell y}dy. \tag{1.27}$$

Thus,

$$h_-(0,k)h_+(0,\ell) = 1 + O(k^{-1}) \qquad \text{as} |k| \to \infty, \ \mathrm{Im}\, k \geq 0$$

$$h_+h'_- = O(k^0); \qquad h_-h'_+ = O(k^0).$$

So from (1.23) we get

$$2ika_-(k) = i(k+\ell)\{1 + O(k^{-1})\} + O(k^0)$$

$$a_-(k) = \frac{1}{2}\left(1 + \frac{\ell}{k}\right)\{1 + O(k^{-1})\} + O(k^{-1}).$$

Since $\ell/k = 1 + O(k^{-2})$, result (i) follows.

Proof (ii). Evaluating the Wronskian at $x = 0$ in (1.13) we get,

$$2ikb_-(k) = i(k - \ell)h_\pm^*(0,k)h_+(0,\ell) + h_\pm^{*\prime}(0,k)h_+(0,\ell) - h_\pm^*(0,k)h_+'(0,\ell). \tag{1.28}$$

Making estimates as in part (i) we get

$$2ikb_-(k) = iO(k^{-1})\{1 + O(k^{-1})\} + O(k^0)$$

whence

$$b_-(k) = O(k^{-1}) \qquad \text{as } k \to \pm\infty,\ k \in \mathbf{R}.$$

Proof (iii). We begin again with (1.28). Even under the weaker hypothesis of (i) we had

$$h_+(0,\ell) = 1 + O(k^{-1}), \qquad h_-(0,k) = 1 + O(k^{-1}),$$
$$\text{and } (k - \ell) = O(k^{-1}) \qquad \text{as } k \to \pm\infty.$$

Thus it will suffice to show that the added hypotheses of (iii) imply

$$h_+'(0,\ell) = O(k^{-1}) \quad \text{and} \quad h_-'(0,\ell) = O(k^{-1}).$$

From (1.7) we find that

$$\partial_x B_-(0,y) = v(y) - \int_{z=0}^{y} v(y - z)B_-(y - z,z)dz.$$

Thus $\partial_y\partial_x B_-(0,y)$ exists almost everywhere away from the jump points of $v(y)$. Applying Lemma 3 in §2 of Deift and Trubowitz [6], one can verify that $\partial_y\partial_x B_-(0,y)$ is in $L^1(\mathbf{R}^-)$. From (1.21) we have

$$h_-'(x,k) = \int_{-\infty}^{0} \partial_x B_-(x,y)e^{-2iky}dy.$$

Setting $x = 0$ and integrating by parts yields

$$h_-'(0,k) = \frac{1}{2ik}\left[\partial_x B_-(0,0) - J_1 - \int_{-\infty}^{0} \partial_y\partial_x B_-(0,y)e^{-2iky}dy\right] \tag{1.29}$$

where J_1 represents the jump terms arising from the discontinuities which $\partial_x B_-(0,y)$ inherits from $v(y)$. It follows that $h_-'(0,k) = O(k^{-1})$ as $k \to \pm\infty$. The analogous result for $h_+'(0,\ell)$ can be obtained similarly working with $v(x) - c^2$ in (1.10).

Proof (iv). Again the starting point is (1.28). Since $(k - \ell) = c^2/2k + O(k^{-3})$, we get

$$i(k - \ell)h_\pm^*(0,k)h_+(0,\ell) = ic^2/2k + O(k^{-2}).$$

It will suffice to show that

$$h_\pm^{*\prime}(0,k)h_+(0,\ell) - h_\pm^*(0,k)h_+' = -ic^2/2k + O(k^{-2}).$$

For this it will suffice to obtain

$$(1.30) \qquad h'_{-}(0,k) = \frac{iv(0)}{2k} + O(k^{-2})$$

$$(1.31) \qquad h'_{+}(0,\ell) = \frac{-i\{v(0) - c^2\}}{2\ell} + O(k^{-2}) = \frac{-i\{v(0) - c^2\}}{2k} + O(k^{-2})$$

since we already have (1.24) and (1.25).

Under the hypotheses of (iv), formula (1.29) yields

$$h'_{-}(0,k) = \frac{1}{2ik}\left[\partial_x B_{-}(0,0) - \int_{-\infty}^{0} \partial_y \partial_x B_{-}(0,y) e^{-2iky} dy\right].$$

The added regularity of v also implies that $\partial_y \partial_x B_{-}(0,y)$ is piecewise absolutely continuous and $\partial_y^2 \partial_x B_{-}(0,y)$ is in $L^1(\mathbf{R}^{-})$. Note that $\partial_x B_{-}(0,0) = v(0)$. Another integration by parts yields

$$h'_{-}(0,k) = \frac{1}{2ik}\left[v(0) - \frac{1}{2ik}\left\{J_2 + \int_{-\infty}^{0} \partial_y^2 \partial_x B_{-}(0,y) e^{-2iky} dy\right\}\right]$$

where J_2 represents the jump terms from $\partial_y \partial_x B_{-}(0,y)$. The decay (1.30) now follows; (1.31) can be verified similarly. □

Corollary 1.5. *$a_{-}(k)$ has at most a finite number of zeros in* $\operatorname{Im} k > 0$ *under either of the additional hypotheses*

(A) $W(0) \neq 0$

or

(B) $W(0) = 0$ *and* $\lim\limits_{\substack{k \to 0 \\ \operatorname{Im} k \geq 0}} W(k)/k = i\gamma$ *with* $0 \neq \gamma \in \mathbf{R}$.

Proof. Since $a_{-}(k) = 1 + O(k^{-1})$ at ∞, there is an R_0 such that $|a_{-}(k)| > 1/2$ for $|k| > R_0$. By Lemma 1.2 and (1.13) it follows that $a_{-}(k) \neq 0$ for nonzero real k.

Let Z denote the set of zeros of $a_{-}(k)$ with $\operatorname{Im} k > 0$. We know

$$Z \subset \{k : |k| \leq R_0, \operatorname{Im} k \geq 0\}.$$

Suppose Z is infinite. Then Z must have an accumulation point k_0. The analyticity and continuity properties of $a_{-}(k)$ force k_0 to be 0. So we now know that $0 = \lim\limits_{n \to \infty} k_n$ for some sequence in Z.

In case (A) we now get the contradiction

$$0 \neq W(0) = \lim_{n \to \infty} W(k_n) = \lim_{n \to \infty} 2ik_n a_{-}(k_n) = 0$$

from (1.13) and the continuity of W at 0.

In case (B) we get the contradiction

$$0 \neq \lim_{k\to 0} \frac{W(k)}{k} = \lim_{n\to\infty} \frac{W(k_n)}{k_n} = \lim_{n\to\infty} 2ia_-(k_n) = 0$$

from (1.13) above. □

Remark. Condition (A) above holds in the generic case for potentials in class $P(c,1)$. In the exceptional case, $W(0) = 0$, the second condition in (B) can be verified for potentials in class $P(c,2)$, i.e. satisfying (0.3) with $N = 2$. See Proposition 2.4 at the end of this section.

The scattering theory is best developed in the case that $a_-(k)$ has only finitely many zeros. Thus we will restrict our attention to potentials for which either (A) or (B) holds.

Definition 1. Suppose v is a potential in class $P(c,1)$ and that either (A) or (B) holds. The "scattering data" of v consists of the following objects:

(i) the *transmission coefficients,* which are given by

$$T_+(k) = \frac{1}{a_+(k)} = \frac{2i\ell}{W[f_-,f_+]} \qquad \text{for } \operatorname{Im} k > 0,\ k \neq 0.$$

$$T_-(k) = \frac{1}{a_-(k)} = \frac{2ik}{W[f_-,f_+]} \qquad \text{for } \operatorname{Im} k > 0,\ k \neq 0.$$

(ii) the *reflection coefficients,* which are given by

$$R_+(k) = \frac{b_+(k)}{a_+(k)} = \frac{W[f_+^*,f_-]}{W[f_-,f_+]} \qquad \text{for } k \in \mathbf{R},\ k \neq 0.$$

$$R_-(k) = \frac{b_-(k)}{a_-(k)} = \frac{W[f_+,f_-^*]}{W[f_-,f_+]} \qquad \text{for } k \in \mathbf{R},\ k \neq 0.$$

(iii) the *poles* of $T_-(k)$, which are a finite set (possibly empty) of pure imaginary numbers $i\kappa_j$, j in index set J. The convention is that if $\mathcal{J} = \#J > 0$, then we take $J = \{1,2,\ldots,\mathcal{J}\}$ and $0 < \kappa_1 < \kappa_2 < \ldots < \kappa_{\mathcal{J}}$. Note that the numbers $-\kappa_j^2$ are the eigenvalues of the operator $-\partial_x^2 + v$. If T_- has no poles we say "v has no bound states."

(iv) the *norming constants,* which are the positive numbers given by

$$c_{\pm j} = \left[\int_{-\infty}^{\infty} |f_\pm(x,i\kappa_j)|^2 dx\right]^{-1}, \qquad \text{for } j \in J.$$

(v) the *dependence constants,* which are the nonzero numbers μ_j for $j \in J$ given by

$$f_-(x,i\kappa_j) = \mu_j f_+(x,i\kappa_j).$$

In discussing the scattering data we make use of the following functions.

Definition 2.

$$F_-(x) = \pi^{-1} \int_{-\infty}^{\infty} R_-(k)\, e^{-2ikx} dk$$

$$G_-(x) = 2 \sum_J c_{-j}\, e^{2\kappa_j x}$$

$$F_+(x) = \pi^{-1} \int_{-\infty}^{\infty} R_+(k(\ell))\, e^{2i\ell x} d\ell$$

$$G_+(x) = 2 \sum_J c_{+j}\, e^{-2\lambda_j x} \qquad \text{where } \lambda_j = \sqrt{c^2 + k_j^2}$$

$$H_+(x) = \pi^{-1} \int_{k=0}^{c} |T_-(k)|^2\, e^{-2\lambda x} dk \qquad \text{where } \lambda = \lambda(k) = \sqrt{c^2 - k^2}$$

$$\Omega_+(x) = F_+(x) + G_+(x) + H_+(x)$$

$$\Omega_-(x) = F_-(x) + G_-(x).$$

§2. Properties of the scattering data.

Theorem 2.1. *Consider a potential v in class $P(c,1)$ such that either*

(A) $W(0) \neq 0$

or

(B) $W(0) = 0$ *and* $\lim_{k \to 0} W(k)/k = i\gamma$ *for* $0 \neq \gamma \in \mathbf{R}$.

Then the scattering data of v must satisfy the following conditions:

(C.1) $T_\pm(-k) = T_\pm^*(k)$ *and* $R_\pm(-k) = R_\pm^*(k)$ *for all k in* $\mathbf{R} \sim \{0\}$.

(C.2) *The functions $R_\pm$ extend to* $\mathbf{R}$. *The functions $T_\pm$ are mermorphic in* $\operatorname{Im} k > 0$ *and continuous on* $\{k : 0 \leq \operatorname{Im} k < \underline{\kappa}\} \sim \{0\}$ *where* $\underline{\kappa} = \inf\{\kappa_j : j \in J\}$. *$T_-$ extends continuously to* 0.

(C.3) *(i) For all k with* $\operatorname{Im} k \geq 0$, $k \neq 0$, $k \notin \{i\kappa_j : j \in J\}$,

$$kT_+(k) = \ell T_-(k).$$

(ii) For real k with $|k| > c$,

$$1 = \frac{k}{\ell} |T_+|^2 + |R_+|^2 = \frac{\ell}{k} |T_-|^2 + |R_-|^2 \qquad 0 = \ell T_- R_\pm^* + kT_\mp^* R_+.$$

(iii) For real k with $0 < |k| \leq c$,

$$R_-(k) = T_-(k)/T_\pm^*(k) \quad \textit{and} \quad R_+(k) = -1.$$

(iv) *If* $\operatorname{Im} k \geq 0$ *and* $k \neq 0$, *then* $T_-(k) \neq 0$.
If $\operatorname{Im} k \geq 0$ *and* $k \notin \{-c,0,c\}$, *then* $T_+(k) \neq 0$.

In case (A) holds, then

(vA) T_+ *extends continuously to* $k = 0$; $T_+(0) \in \mathbf{R} \sim \{0\}$; $T_+(\pm c) = 0$; $T_-(0) = 0$; $\dot{T}_-(0) \equiv \lim_{k\to 0} T_-(k)/k = i\alpha$ *for* $0 \neq \alpha \in \mathbf{R}$.

(viA) $\mathbf{R}_-(0) = -1$.

In case (B) holds, then

(vB) $T_+(k) = O(k^{-1})$ *as* $k \to 0$; $T_+(\pm c) = 0$; $T_-(0) \neq 0$.

(viB) $R_-(0) = +1$.

(C.4) $T_\pm(k) = 1 + 0(k^{-1})$ *as* $|k| \to \infty$ *in* $\operatorname{Im} k \geq 0$
$R_\pm(k) = O(k^{-1})$ *as* $k \to \pm\infty$, $k \in \mathbf{R}$,

(C.5) *The poles of* T_+ *in* $\operatorname{Im} k > 0$ *are the same as the poles of* T_-, *namely the points* $i\kappa_j$ *for* $j \in J$. *These poles are all simple and*

$$\operatorname{Res}(T_-, i\kappa_j) = \operatorname{sgn}(\mu_j)\, i\sqrt{c_{+j}c_{-j}}.$$

(C.6) *The functions* F_+ *and* F_- *are absolutely continuous. Further,*

$$\int_{-\infty}^{X} |F'_-(x)|(1 + |x|)dx < \infty \quad \textit{and} \quad \int_X^\infty |F'_+(x)|(1 + |x|)dx < \infty$$

for all finite X.

Proof. Conditions (C1) and (C2) follow from Lemmas 1.1, 1.2, 1.3, and 1.4. The proof of continuity of T_- at 0 is immediate in case (A); in case (B) note that

$$\lim_{k\to 0} T_-(k) = \lim_{k\to 0} \frac{2ik}{W(k)} = 2i\left(\lim_{k\to 0} \frac{W(k)}{k}\right)^{-1}.$$

From (C3i) it will follow that $T_+(k) = O(k^{-1})$ at 0. The continuity of $R_\pm$ at 0 is discussed below.

Parts (i), (ii), and (iii) of (C3) follow from Lemma 1.1. Part (iv) follows from the definitions of $T_\pm$. We discuss the last two parts separately in cases (A) and (B).

Suppose that $W(0) \neq 0$. Then by definition $T_-(0) = \lim_{k\to 0} 2ik/W(k) = 0$ and $T_+(\pm c) = 0$ by part (i). Next note that

$$\frac{T_-(k) - T_-(0)}{k - 0} = \frac{T_-(k)}{k} = \frac{2i}{W(k)} \to \frac{2i}{W(0)} \qquad \text{as } k \to 0.$$

Since both $f_+(x,0)$ and $f_-(x,0)$ are real valued, $W(0) \in \mathbf{R}$. Thus,

$$\dot{T}_-(0) = \lim_{k\to 0} T_-(k)/k = \frac{2i}{W(0)} = i\alpha \qquad \text{with } \alpha = \frac{2}{W(0)} \in R \sim \{0\}.$$

Thus (vA) is proved. We next look at (viA). For $k \in \mathbf{R}$, $|k| \leq c$ we know that

$f_+(x,k)$ is real valued, thus

$$R_+(k) = \frac{W[f_+^*,f_-]}{W[f_-,f_+]} = \frac{W[f_+,f_-]}{W[f_-,f_+]} = -1$$

for $-c \le k \le +c$. By part (iii) and (vA) we have.

$$R_-(0) = \lim_{k\to 0} R_-(k) = \lim_{k\to 0} T_-(k)/T_-^*(k) = \lim_{k\to 0} \frac{0 + i\alpha k + o(k)}{0 - i\alpha k + o(k)} = -1.$$

Now we turn to case (B). As we saw $T_-(0)$ is the limit as $k \to 0$ of $2ik/W(k) = 2/\gamma \ne 0$. For R_- note that if $0 \ne k \in \mathbf{R}$ we get

$$R_-(k) = \frac{T_-(k)}{T_-^*(k)} = \frac{2ik}{W(k)}\left(\frac{W(k)}{2ik}\right)^* = \frac{-W^*(k)}{W(k)}$$

$$= -\frac{0 - i\gamma k + o(k)}{0 + i\gamma k + o(k)} \to +1 \qquad \text{as } k \to 0.$$

But by part (3.i), $T_+(\pm c) = 0$. The derivatives $\dot{T}_+(\pm c)$ still cannot exist. This completes (C3).

(C4) follows from Lemma 1.4 and its corollary.

For (C5) recall that $T_-(k) = 1/a_-(k) = 2ik/W(k)$ and $T_+(k) = 2i\ell/W(k)$. Thus the poles of T_+ in the upper half-plane are the zeros of $W(k)$, which are the poles of $T_-(k)$. Since

$$\dot{a}_-(i\kappa_j) = -i \int_{-\infty}^{\infty} f_+(x,i\kappa_j) f_-(x,i\kappa_j) dx \ne 0$$

by Lemma 1.3, the poles of T_- are simple and

$$\operatorname{Res}(T_-,i\kappa_j) = 1/\dot{a}_-(i\kappa_j).$$

By the definition of $c_{\pm j}$ and μ_j we have

$$\mu_j^2 = c_{+j}/c_{-j} \quad \text{or} \quad \mu_j = \operatorname{sgn}(\mu_j)\sqrt{c_{+j}/c_{-j}}.$$

Now we see that

$$\dot{a}_-(i\kappa_j) = -i\mu_j^{-1} \int_{-\infty}^{\infty} f_-(x,i\kappa_j)^2 dx = -i\mu_j^{-1}(c_{-j})^{-1}$$

whence

$$\operatorname{Res}(T_-,i\kappa_j) = i\mu_j c_{-j} = ic_{+j}/\mu_j = i \operatorname{sgn}(\mu_j)\sqrt{c_{+j}c_{-j}}.$$

Since $T_+(k) = \ell T_-(k)/k$ we note here also that

$$\operatorname{Res}(T_+,i\kappa_j) = \frac{\ell(i\kappa_j)}{i\kappa_j}\operatorname{Res}(T_-,i\kappa_j) = \frac{\lambda_j \operatorname{Res}(T_-,i\kappa_j)}{\kappa_j}$$

where λ_j is, as before, $\sqrt{\kappa_j^2 + c^2}$.

The proof of (C6) requires the Marchenko equations and is postponed until after Lemma 2.2, in which these equations are derived.

Lemma 2.2. *Suppose v is a potential in class $P(c,1)$ such that either (A) or (B) holds. Then for all $x \in \mathbf{R}$ and $y \leq 0$*

$$B_-(x,y) + \Omega_-(x,y) + \int_{-\infty}^{0} B_-(x,z)\Omega_-(x + y + z)dz = 0. \tag{2.1}$$

Also, for all x in $\mathbf{R}$ and all $y \geq 0$

$$B_+(x,y) + \Omega_+(x + y) + \int_{0}^{\infty} B_+(x,z)\Omega_+(x + y + z)dz = 0. \tag{2.2}$$

Proof. We derive (2.1) first, starting with the relation

$$T_-(k)f_+(x,k) = f_-^*(x,k) + R_-(k)f_-(x,k),$$

which is valid for all nonzero real k and any fixed x in $\mathbf{R}$.

In case (A) we know that R_- is continuous and $O(k^{-1})$ at ∞, so $R_- \in L^2(\mathbf{R})$. In case (B) we know that R_- is continuous on $\mathbf{R} \sim \{0\}$, that $|R_-(k)| \leq 1$ for $0 \neq k \in \mathbf{R}$, and that $R_-(k) = O(k^{-1})$ at ∞. So in case (B), also, R_- is in $L^2(\mathbf{R})$. Thus

$$R_-(k) = \int_{-\infty}^{\infty} F_-(s)\, e^{2iks}ds$$

in L^2 sense and in Ceasaro mean almost everywhere. Recall that $B_-(x,y) = 0$ for $y > 0$. Now

$$\begin{aligned}
f_-^* + R_-f_- &= \left[e^{ikx}\left\{1 + \int_{-\infty}^{\infty} B_-(x,y)\, e^{2iky}dy\right\}\right] \\
&\quad + \int_{-\infty}^{\infty} F_-(s)\, e^{2iks}ds \left[e^{-ikx}\left\{1 + \int_{-\infty}^{\infty} B_-(x,t)\, e^{-2ikt}dt\right\}\right] \\
&= e^{ikx}\left\{1 + \int_{-\infty}^{\infty} B_-(x,y)\, e^{2iky}dy + \int_{-\infty}^{\infty} F_-(s)\, e^{2ik(s-x)}ds\right. \\
&\qquad \left. + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_-(s)B_-(x,t)\, e^{2ik(s-x-t)}dtds\right\} \\
&= e^{ikx}\left\{1 + \int_{y=-\infty}^{\infty} \mathcal{B}_-(x,y)\, e^{2iky}dy\right\}
\end{aligned}$$

where

$$\mathcal{B}_-(x,y) \equiv B_-(x,y) + F_-(x + y) + \int_{-\infty}^{\infty} F_-(x + y + z)B_-(x,z)dz. \tag{2.3}$$

Thus

$$e^{-ikx}T_-(k)f_+(x,k) - 1 = \int_{-\infty}^{\infty} \mathcal{B}_-(x,y)\, e^{2iky}dy$$

and

$$\mathcal{B}_-(x,y) = \pi^{-1}\int_{-\infty}^{\infty} \{e^{-ikx}T_-(k)f_+(x,k) - 1\}\, e^{-2iky}dk. \tag{2.4}$$

For fixed (x,y) consider the function

$$\phi(k) \equiv \{e^{-ikx}T_-(k)f_+(x,k) - 1\}\, e^{-2iky}.$$

Clearly $\phi(k)$ is meromorphic in $\operatorname{Im} k > 0$. We will also show that if $y \leq 0$, then

$$|\phi(k)| = O(k^{-1}) \qquad \text{as } |k| \to \infty,\ \operatorname{Im} k \geq 0 \tag{2.5}$$

and

$$|\phi(k)| = O(e^{2y\operatorname{Im} k}) \qquad \text{as } \operatorname{Im} k \to +\infty. \tag{2.6}$$

First note that $|e^{-2iky}| = e^{2\operatorname{Im} ky}$. Then make the decomposition

$$\begin{aligned} e^{-ikx}T_-(k)f_+(x,k) - 1 &= e^{-ikx}T_-(k)\, e^{i\ell x}\{1 + h_+(x,\ell)\} - 1 \\ &= e^{i(\ell-k)x}\{T_-(k) - 1\} + \{e^{i(\ell-k)x} - 1\} + e^{i(\ell-k)x}T_-(k)h_+(x,\ell). \end{aligned}$$

Since $\ell - k = O(k^{-1})$ it follows that $e^{i(\ell-k)x} - 1 = O(k^{-1})$. We already know that $T_-(k) - 1 = O(k^{-1})$ and $h_+(x,\ell) = O(k^{-1})$. Thus

$$e^{-ikx}T_-(k)f_+(x,k) - 1 = O(k^{-1}).$$

Since $y \leq 0$, (2.5) and (2.6) follow.

The two decay rates (2.5) and (2.6) mean that if $y < 0$, then we can evaluate the integral in (2.4) by closing the contour in the upper half-plane. This yields

$$\begin{aligned} \mathcal{B}_-(x,y) &= \pi^{-1}2\pi i \sum \{\text{residues of } \phi(k) \text{ in } \operatorname{Im} k > 0\} \\ &= 2i\sum_J \operatorname{Res}(e^{-ikx}T_-(k)f_+(x,k)\, e^{-2iky}, i\kappa_j) \\ &= 2i\sum_J e^{\kappa_j x}f_+(x,i\kappa_j)\, e^{+2\kappa_j y}\operatorname{Res}(T_-, i\kappa_j) \\ &= 2i\sum_J e^{\kappa_j x}\mu_j^{-1}f_-(x,i\kappa_j)\, e^{2\kappa_j y}\operatorname{sgn}(\mu_j)\, i\sqrt{c_{+j}c_{-j}} \end{aligned}$$

by (C5). Recalling further that $|\mu_j| = \sqrt{c_{+j}/c_{-j}}$,

$$\mathcal{B}_-(x,y) = -2\sum_J e^{2\kappa_j x}\{1 + h_-(x,i\kappa_j)\}\, e^{2\kappa_j y}c_{-j} \tag{2.7}$$

$$= -2 \sum_J c_{-j}\, e^{2\kappa_j(x+y)} \left\{ 1 + \int_{-\infty}^{0} B_-(x,z)\, e^{2\kappa_j z} dz \right\}$$

$$= -2 \sum_J c_{-j}\, e^{2\kappa_j(x+y)} - 2 \int_{-\infty}^{0} B_-(x,z) \sum_{j=1}^{N} c_{-j}\, e^{2\kappa_j(x+y+z)} dz.$$

With $\Omega_-(x)$ defined by Definition 2 we now combine (2.3) and (2.7) to obtain the left-hand Marchenko equation

$$B_-(x,y) + \Omega_-(x,y) + \int_{-\infty}^{0} B_-(x,z)\Omega_-(x + y + z) dz = 0,$$

for $x \in \mathbf{R}$ and $y < 0$. By continuity it holds also at $y = 0$.

To prove (1.32) we begin with

$$T_+(k)\, f_-(x,k) = f_+^*(x,k) + R_+(k) f_+(x,k)$$

which is valid for all $x \in \mathbf{R}$ provided $k \in \mathbf{R}$ and $|k| > c$, by (1.11) and the definition of T_+ and R_+. Now

$$R_+(k(\ell)) = \int_{-\infty}^{\infty} F_+(x)\, e^{2i\ell x} dx \qquad \text{for } \ell \in \mathbf{R}.$$

So for $k = k(\ell)$ with $\ell \in \mathbf{R}$, we get

$$f_+^* + R_+ f_+ = e^{-i\ell x} \left\{ 1 + \int_{-\infty}^{\infty} \mathcal{B}_+(x,y)\, e^{-2i\ell y} dy \right\}$$

where

$$\mathcal{B}_+(x,y) = B_+(x,y) + F_+(x + y) + \int_0^{\infty} B_+(x,z) F_+(x + y + z) dz. \tag{2.8}$$

Note that $B_+(x,\cdot) \in L^1$ and $F_+ \in L^2$ so the last term in $\mathcal{B}_+(x,\cdot)$ is essentially the convolution of an L^2 function by an L^1 function, and is therefore in L^2. Now we get—for $\ell \in \mathbf{R}$—

$$e^{i\ell x} T_+(k(\ell)) f_-(x,k(\ell)) - 1 = \int_{-\infty}^{\infty} \mathcal{B}_+(x,y)\, e^{-2i\ell y} dy.$$

So

$$\mathcal{B}_+(x,y) = \pi^{-1} \int_{\ell=-\infty}^{\infty} \{ e^{i\ell x} T_+(k(\ell)) f_-(x,k(\ell)) - 1 \}\, e^{2i\ell y} d\ell.$$

The integrand is not analytic as a function of ℓ in $\operatorname{Im} \ell > 0$, since $k(\ell)$ is not analytic in the full upper half-ℓ-plane. Therefore we change variables in the real axis, letting $\ell = \ell(k)$ for $k \in \mathbf{R}$, $|k| > c$.

$$(2.9) \qquad \mathcal{B}_+(x,y) = \pi^{-1} \int_{k=-\infty}^{\infty} \{e^{i\ell x} T_+(k) f_-(x,k) - 1\} e^{2i\ell y} k\ell^{-1} dk$$
$$- \pi^{-1} \int_{k=-c}^{c} \{e^{i\ell x} T_+(k) f_-(x,k) - 1\} e^{2i\ell y} k\ell^{-1} dk.$$

We now need to consider the integrand

$$\psi(k) = \{e^{i\ell x} T_+(k) f_-(x,k) - 1\} e^{2i\ell y} k/\ell$$

for fixed x and y. Clearly $\psi(k)$ is meromorphic for $\operatorname{Im} k > 0$. If $y > 0$, and $\operatorname{Im} k \geq 0$, then $|e^{2i\ell y}| = e^{-2\operatorname{Im}\ell y} \leq 1$. As in part (i) we can now show that

$$|\psi(k)| = O(k^{-1}) \qquad \text{as } |k| \to \infty \text{ with } \operatorname{Im} k \geq 0$$

and

$$|\psi(k)| = O(e^{-2y\operatorname{Im}\ell}) \qquad \text{as } \operatorname{Im} k \uparrow +\infty.$$

Therefore, for x fixed in $\mathbf{R}$ and y fixed positive, we can evaluate $\int_{\mathbf{R}} \psi(k)dk$ by closing the contour in the upper half-plane.

$$\pi^{-1} \int_{k=-\infty}^{\infty} \psi(k)dk = 2i \sum \{\text{Residues of } \psi \text{ in } \operatorname{Im} k > 0\}$$
$$= 2i \sum_J \operatorname{Res}(\{e^{i\ell x} T_+ f_- - 1\} e^{2i\ell y} k/\ell, i\kappa_j)$$
$$= 2i \sum_J e^{-\lambda_j x} f_-(x,i\kappa_j) e^{-2\lambda_j y} \kappa_j \lambda_j^{-1} \operatorname{Res}(T_+, i\kappa_j)$$

where $\lambda_j = \sqrt{\kappa_j^2 + c^2}$. Recall that

$$f_-(x,i\kappa_j) = \mu_j f_+(x,i\kappa_j) = \mu_j e^{-\lambda_j x}\{1 + h_+(x,i\lambda_j)\}$$

and

$$\operatorname{Res}(T_+, i\kappa_j) = \frac{\lambda_j}{\kappa_j} \operatorname{Res}(T_-, i\kappa_j) = \frac{\lambda_j}{\kappa_j} \operatorname{sgn}(\mu_j) i \sqrt{c_{+j} c_{-j}}$$

and

$$|\mu_j| = \sqrt{c_{+j}/c_{-j}}.$$

Thus

$$(2.10) \qquad \pi^{-1} \int_{k \in \mathbf{R}} \psi(k)dk = 2i \sum_J \mu_j e^{-2\lambda_j x}\{1 + h_+\} e^{-2\lambda_j y} \operatorname{sgn}(\mu_j) i \sqrt{c_{+j} c_{-j}}$$
$$= -2 \sum_J \{1 + h_+\} e^{-2\lambda_j (x+y)} c_{+j}$$

$$= -2 \sum_J c_{+j}\, e^{-2\lambda_j(x+y)} \left\{ 1 + \int_0^\infty B_+(x,z)\, e^{-2\lambda_j z} dz \right\}$$

$$= \left[-2 \sum_J c_{+j}\, e^{-2\lambda_j(x+y)} \right]$$

$$+ \int_0^\infty B_+(x,z) \left[-2 \sum_J c_{+j}\, e^{-2\lambda_j(x+y+z)} \right] dz.$$

We must now look at the integral of $\psi(k)$ on $-c < k < c$.

$$\pi^{-1} \int_{-c}^{c} \psi(k) dk = \pi^{-1} \int_{-c}^{c} \{ e^{i\ell(k)x} T_+(k) f_-(x,k) - 1 \}\, e^{2i\ell y} (k/\ell) dk$$

$$= \pi^{-1} \int_{-c}^{c} \{ e^{-\lambda(k)x} T_+(k) f_-(x,k) - 1 \} e^{-2\lambda(k)y} \frac{k}{i\lambda(k)}\, dk$$

where $\lambda(k) = \sqrt{c^2 - k^2}$ for $|k| \le c$. Note that the singularities at $k = \pm c$ are of the order of $1/\sqrt{k - (\pm c)}$, and are thus integrable. Make the change of variable $k' = -k$ in the piece of the integral over $[-c,0]$:

$$\int_{-c}^{0} \{ e^{-\lambda(k)x} T_+(k) f_-(x,k) - 1 \}\, e^{-2\lambda y} k (i\lambda)^{-1} dk$$

$$= -\int_{k'=c}^{0} \{ e^{-\lambda(k')x} T_+(-k') f_-(x,-k') - 1 \}\, e^{-2\lambda(k')y} \frac{(-k')dk'}{i\lambda(k')}$$

$$= -\int_{k=0}^{c} \{ e^{-\lambda(k)x} T_+^*(k) f_-^*(x,k) - 1 \} e^{-2\lambda(k)y} \frac{k}{i\lambda(k)}\, dk.$$

Thus

$$\pi^{-1} \int_{-c}^{c} \psi(k) dk = \pi^{-1} \int_{k=0}^{c} e^{-\lambda(k)x} \{ T_+(k) f_-(x,k) - T_+^*(k) f_-^*(x,k) \}\, e^{-2\lambda(k)y} \frac{k}{i\lambda}\, dk.$$

Now

$$\begin{aligned} kT_+f_- - kT_+^*f_-^* &= \ell T_-f_- - (\ell T_-)^* f_-^* \\ &= i\lambda T_-f_- + i\lambda T_-^* f_-^* \\ &= i\lambda T_-^*(f_-^* + T_-f_-/T_-^*) \\ &= i\lambda T_-^*(f_-^* + R_-f_-) \qquad \text{by (C3)} \\ &= i\lambda T_-^* T_- f_+ \qquad \text{by (1.10).} \end{aligned}$$

So

$$
\begin{aligned}
(2.11)\qquad \pi^{-1}\int_{-c}^{c}\psi(k)dk &= \pi^{-1}\int_{0}^{c} e^{-\lambda(k)x}\,|T_-(k)|^2 f_+(x,k)\,e^{-2\lambda(k)y}dk \\
&= \pi^{-1}\int_{0}^{c} |T_-(k)|^2\,e^{-2\lambda(k)(x+y)}\{1 + h_+(x,i\lambda)\}dk \\
&= \pi^{-1}\int_{0}^{c} |T_-(k)|^2\,e^{-2\lambda(x+y)}dk \\
&\quad + \int_{0}^{\infty} B_+(x,z)\pi^{-1}\int_{0}^{c} |T_-(k)|^2 e^{-2\lambda(x+y+z)}dkdz \\
&= H_+(x+y) + \int_{0}^{\infty} B_+(x,z)H_+(x+y+z)dz.
\end{aligned}
$$

Therefore from (2.9) through (2.11) we conclude that

$$
\begin{aligned}
B_+(x,y) &+ F_+(x+y) + \int_{0}^{\infty} B_+(x,z)F_+(x+y+z)dz \\
&= \pi^{-1}\int_{-\infty}^{\infty}\psi(k)dk - \pi^{-1}\int_{-c}^{c}\psi(k)dk \\
&= \left[-2\sum_J c_{+j}e^{-2\lambda j(x+y)}\right] + \int_{0}^{\infty} B_+(x,z)\left[-2\sum_J c_{+j}e^{-2\lambda j(x+y+z)}\right]dz \\
&\quad - H_+(x+y) - \int_{0}^{\infty} B_+(x,z)H_+(x+y+z)dz
\end{aligned}
$$

whence

$$B_+(x,y) + \Omega_+(x+y) + \int B_+(x,z)\Omega_+(x+y+z)dz = 0 \qquad \text{for } y > 0.$$

By continuity, this also holds for $y = 0$. □

Remark. A condition on v, weaker than (A) or (B), which still suffices for this argument is the condition that

$$\lim_{\eta\downarrow 0} T_-(\xi + i\eta)\chi_{[-c,c]}(\xi) = T_-(\xi)\chi_{[-c,c]}(\xi) \qquad \text{in } L^2([-c,c]).$$

We can now return to the proof of (C6) in Theorem 2.1. Actually we will prove a stronger result including (C6).

Lemma 2.3. *Suppose $v \in P(c,N)$ for $N \geq 1$. If $N = 1$ assume also that either condition (A) or (B) holds. Then the functions F_+ and F_- are absolutely continuous. Further for all finite X*

$$\int_{-\infty}^{X} |F'_{-}(x)|(1 + |x|^N)dx < \infty$$

and

$$\int_{X}^{\infty} |F'_{+}(x) + H'_{+}(x)|(1 + |x|^N)dx < \infty.$$

Proof. Consider the Marchenko equation (2.2) as an integral equation for Ω_+ in terms of B_+. For simplicity set $y = 0^+$. Thus

$$\Omega_+(x) + \int_0^{\infty} B_+(x,z)\Omega_+(x + z)dz = -B_+(x,0). \tag{2.12}$$

From (2.12) and the properties of $B_\pm(x,y)$, it follows that Ω_+ is absolutely continuous. Further one obtains the bounds

$$|\Omega_+(x)| \leq C_1(x) \int_x^{\infty} |v(s) - c^2|ds \tag{2.13a}$$

and

$$|\Omega'_+(x) - \{v(x) - c^2\}| \leq C_2(x)\left[\int_x^{\infty} |v(s) - c^2|ds\right]^2 \tag{2.13b}$$

where $C_1(x)$ and $C_2(x)$ are positive nonincreasing functions of x.

We now pick $X \in \mathbf{R}$ and show that

$$\int_X^{+\infty} |\Omega'_+(x)|(1 + |x|^N)dx < \infty.$$

By (2.13b) we see that

$$\sup_{x \geq X}|\Omega'(x)| \leq \sup_{x \geq X}|v(x) - c^2| + C_2(X)\left[\int_X^{\infty} |v(s) - c^2|ds\right]^2 = K(X) < \infty.$$

Thus it suffices to show that

$$\int_0^{\infty} |\Omega'_+(x)|(1 + x^N)dx < \infty$$

since, if $X < 0$, the integrand will be bounded on $[X,0]$. Clearly

$$\int_0^{\infty} |\Omega'_+(x)|(1 + x^N)dx \leq \int_0^{\infty} |v(x) - c^2|(1 + x^N)dx$$
$$+ C_2(X) \int_0^{\infty} \left[\int_x^{\infty} |v(s) - c^2|ds\right]^2 (1 + x^N)dx$$

the first integral is finite by (0.3). Since $0 \le x \le s$ in the second integral, I_2, we have

$$\int_x^\infty |v(s) - c^2| ds \le \frac{1}{1 + x^N} \int_x^\infty |v(s) - c^2|(1 + s^N) ds$$

and therefore

$$\begin{aligned} I_2 &= \int_{x=0}^\infty \left[\int_{s=x}^\infty |v(s) - c^2| ds \right]^2 (1 + x^N) dx \\ &\le \int_{x=0}^\infty \left[(1 + x^N)^{-1} \int_{s=x}^\infty |v(s) - c^2|(1 + s^N) ds \right]^2 (1 + x^N) dx \\ &\le \int_{x=0}^\infty (1 + x^N)^{-1} \left[\int_{s=0}^\infty |v(s) - c^2|(1 + s^N) ds \right]^2 dx \\ &= \int_{x=0}^\infty (1 + x^N)^{-1} dx \left[\int_{s=0}^\infty |v(s) - c^2|(1 + s^N) ds \right]^2 < \infty. \end{aligned}$$

Now recall that

$$F_+(x) = \Omega_+(x) - G_+(x) - H_+(x) = \Omega_+(x) - 2 \sum c_{+j} e^{-2\lambda_j x} - \pi^{-1} \int_0^c |T_-(k)|^2 e^{-2\lambda s} dk$$

where $\lambda = \lambda(k) = (c^2 - k^2)^{1/2}$. Clearly G_+ is smooth. Since $v \in P(c,N)$ with $N \ge 2$, $T_-(k)$ is continuous on $[0,c]$ so it is straightforward to verify that $H_+(x)$ is everywhere differentiable. Thus F_+ is absolutely continuous. Since $G'_+(x)$ decays exponentially at $+\infty$, it is also clear that

$$\int_0^\infty |F'_+(x) + H'_+(x)|(1 + |x|^N) dx < \infty.$$

The argument for F'_- is analogous, but simpler since there is no "H" term:

$$F_-(x) = \Omega_-(x) - G_-(x) = \Omega_-(x) - \sum c_{-j} e^{2\kappa_j x}. \qquad \square$$

Remarks. Consider the case $v(x) = H(x)$, the Heavyside function itself. $H \in P(1,N)$ for all $N \ge 1$. One can compute R_+ and T_- explicitly and show that even though the sum $F'_+(x) + H'_+(x)$ is integrable on $[0,\infty)$ with respect to $(1 + x^2)dx$, neither $F'(x)$ nor $H'(x)$ alone is integrable with respect to $(1 + x^2)dx$ on $[0,\infty)$. See [5] for explicit computations.

Proposition 2.4. *Suppose $v \in P(c,2)$. Then either*

(A) $W(0) \ne 0$, *or*

(B) $W(0) = 0$ *and* $\dot{W}(0) \equiv \lim\limits_{\substack{k\to 0 \\ \operatorname{Im} k \geq 0}} \dfrac{W(k)}{k} = i\gamma$ *for* $0 \neq \gamma \in \mathbf{R}$.

Proof. Let us suppose $W(0) = 0$. We need to show first that $\dot{W}(0)$ exists and is pure imaginary and second that $\dot{W}(0) \neq 0$. We use dot ($\dot{}$) for k-derivatives and prime ($'$) for x-derivatives.

Since $W(0) = W[f_-(x,0), f_+(x,0)] = 0$, there is a constant μ_0 such that

$$f_-(x,0) = \mu_0 f_+(x,0). \tag{2.14}$$

Since both $f_\pm(x,0)$ are real and nonzero, we see $0 \neq \mu_0 \in \mathbf{R}$.

Recall that

$$W(k) = f_-(x,k)f_+'(x,k) - f_+(x,k)f_-'(x,k).$$

So formally

$$W^{\cdot}(0) = f_-^{\cdot}(x,0)f_+'(x,0) + f_-(x,0)f_+^{\cdot\prime}(x,0) - f_+^{\cdot}(x,0)\, f_-'(x,0) - f_+(x,0)f_-^{\cdot\prime}(x,0).$$

By (2.14) this becomes

$$\begin{aligned} W(0) = {} & f_-^{\cdot}(x,0)\mu_0^{-1}f_-'(x,0) + f_-(x,0)f_+^{\cdot\prime}(x,0) \\ & - f_+^{\cdot}(x,0)f_-'(x,0) - \mu_0^{-1}f_-(x,0)f_-^{\cdot\prime}(x,0). \end{aligned}$$

We already know that

$$f_-(x,0) = h_-(x,0) = 1 + \int_{-\infty}^{0} B_-(x,y)dy \in \mathbf{R}$$

$$f_-'(x,0) = h_-'(x,0) = \int_{-\infty}^{0} \partial_x B_-(x,y)dy \in \mathbf{R}.$$

One also finds that

$$f_-^{\cdot}(x,0) = -ix\left\{1 + \int_{-\infty}^{0} B_-(x,y)dy\right\} - 2i\int_{-\infty}^{0} yB_-(x,y)dy \in i\mathbf{R}$$

and

$$\begin{aligned} f_-^{\cdot\prime}(x,0) = -i\left\{1 + \int_{-\infty}^{0} B_-(x,y)dy\right\} - ix\int_{-\infty}^{0} \partial_x B_-(x,y)dy \\ - 2i\int_{-\infty}^{0} y\partial_x B(x,y)dy \in i\mathbf{R}. \end{aligned}$$

At this point we use the added decay of v at $-\infty$ given by (0.3) with $N = 2$ to ensure that the integrals above are finite.

We next look at $\dot{f}_+(x,0)$ and $\dot{f}'_+(x,0)$. By the chain rule

$$\dot{f}_+(x,0) = \frac{\partial}{\partial \ell}[f_+(x,\ell)]\Big|_{\ell=ic} \cdot \frac{\partial \ell}{\partial k}\Big|_{k=0}$$

$$\dot{f}'_+(x,0) = \frac{\partial}{\partial \ell}[f'(x,\ell)]\Big|_{\ell=ic} \cdot \frac{\partial \ell}{\partial k}\Big|_{k=0}.$$

Now $\partial\ell/\partial k = k/\ell$, which is 0 when $k = 0$. So we get $\dot{f}_+(x,0) = \dot{f}'_+(x,0) = 0$ provided we can show that the ℓ-derivatives are finite. But since $v \in P(c,2)$, we know that for $\alpha + \beta \le 1$

$$\int_0^\infty |\partial_x\alpha\partial_y\beta B_+(x,y)|(1 + |y|)dy < \infty$$

and therefore the ℓ-derivatives of f_+ and f'_+ at $\ell = ic$ are finite. We now know that $\dot{W}(0)$ exists and

$$\begin{aligned}\dot{W}(0) &= \mu_0^{-1}\dot{f}_-(x,0)f'_-(x,0) - \mu_0^{-1}f_-(x,0)\dot{f}'_-(x,0)\\ &= \mu_0^{-1}[\dot{f}_-(x,0)f'_-(x,0) - f_-(x,0)\dot{f}'_-(x,0)]\\ &= \mu_0^{-1}W[\dot{f}_-(x,0), f_-(x,0)].\end{aligned}$$

But since $f_-(x,0)$ is real valued and $\dot{f}_-(x,0)$ is pure imaginary it follows that

$$\dot{W}(0) = i\gamma \qquad \text{for } \gamma \in \mathbf{R}.$$

It now remains to show that $\gamma \ne 0$. Recall that $2ika_-(k) = W(k)$ for $0 \ne k$. So we can now extend $a_-(k) = W(k)/2ik$ continuously to $k = 0$ with

$$a_-(0) = \dot{W}(0)/2i = \gamma/2.$$

We also know for $0 \ne k \in \mathbf{R}$, that

$$f_+(x,k) = a_-(k)f^*_-(x,k) + b_-(k)f_-(x,k).$$

But for $k \in \mathbf{R}$, $0 < |k| < c$ we get $b_-(k) = a^*_-(k)$ and

$$f_+(x,k) = a_-f^*_- + a^*_-f_- = 2\,\mathrm{Re}[a_-(k)f^*_-(x,k)].$$

Taking the limit as $k \to 0$ we get

$$f_+(x,0) = 2\,\mathrm{Re}\left[\frac{\gamma}{2}f^*_-(x,0)\right].$$

But since $\gamma \in R$ and $f_-(x,0) \in \mathbf{R}$ we conclude that

$$f_+(x,0) = \gamma f_-(x,0).$$

Since $f_+(x,0)$ cannot vanish for all x, γ cannot be 0.

Proposition 2.5. *Consider a generic potential v in $P(c,1)$ such that*

(2.15) $$\int_{-\infty}^{X} |v(x)|(1 + x^2)dx < \infty \qquad \textit{for all finite } X.$$

Then R_- is differentiable at $k = 0$.

Proof. For real k, $R_-(k) = W[f_+,f_-^*]/W[f_-,f_+]$. For generic v, $R_-(0) = -1$. For real k with $|k| < c$, $f_+(x,k)$ is real. Now for real k with $0 \neq |k| < c$

$$\frac{R_-(k) - R_-(0)}{k} = \frac{1}{k}\left(\frac{W[f_+,f_-^*]}{W[f_-,f_+]} + 1\right)$$
$$= \frac{2i}{W[f_-,f_+]} \cdot \frac{\operatorname{Im} W[f_-,f_+]}{k}.$$

So it is enough to show that $\operatorname{Im} W[f_-,f_+]/k$ has a limit as $k \to 0$. Note that

$$\frac{\operatorname{Im} W[f_-,f_+]}{k} = \operatorname{Im}(f_-f'_+ - f_+f'_-)/k = f'_+\frac{\operatorname{Im} f_-}{k} - f_+\frac{\operatorname{Im} f'_-}{k}.$$

Since $v \in P(c,1)$, f_+ and f'_+ are continuous in $\operatorname{Im} k \geq 0$. Note f_- is real at $k = 0$. So it suffices to show that f_- and f'_- are differentiable at $k = 0$. By Lemma 1, p. 130 of [6] f_- is continuously differentiable with respect to k on $\operatorname{Im} k \geq 0$. Now

$$f'_-(x,k) = -ikf_-(x,k) + e^{-ikx}\int_{-\infty}^{0} \partial_x B_-(x,y)e^{-2iky}dy.$$

The only issue is the differentiability of the integral with respect to k at $k = 0$. This follows from the fact that $y\partial_x B_-(x,y)$ is in $L^1(-\infty,0)$ as a function of y, which is true because of the additional decay (2.15) of v at $-\infty$. □

For convenience of reference we now consolidate our results for the case of potentials v in $P(c,2)$.

Theorem 2.6. *Suppose $v \in P(c,2)$. Then the scattering data have the properties (D.1)–(D.7) where (D.1)–(D.5) are the same as (C1)–(C5) and (D.6), (D.7) are given below:*

(D.6) The functions F_+ and F_- are absolutely continuous. Further,

$$\int_{X}^{\infty} |F'_+(x) + H'_+(x)|(1 + x^2)dx < \infty$$

and

$$\int_{-\infty}^{X} |F'_-(x)|(1 + x^2)dx < \infty$$

for all finite X.

(D.7) In the generic case $R_-(k)$ is differentiable at $k = 0$.

Section 3. Examples. We conclude this chapter with two examples to show that neither case (A) nor (B) is vacuous.

Example 1. If $q \geq 0$ and $q \in P(c,1)$, then $W(0) \neq 0$.

If suffices to show that $f_+(x,0)$ is not a scalar multiple of $f_-(x,0)$. We know that $f_+(x,0) \sim e^{i\ell(0)x} = e^{-cx}$ as $x \to +\infty$. So it will suffice to show that $f_-(x,0)$ does not decay to 0 as $x \to +\infty$. Now

$$f_-(x,0) = h_-(x,0) = 1 + \int_{-\infty}^{0} B_-(x,z)dz.$$

So $f_-(x,0)$ is real. By [6, §2] we have

$$h_-(x,0) = 1 + \int_{-\infty}^{x} (x - t)q(t)h_-(t,0)dt.$$

Solving this by iteration we find that $h_-(x,0) \geq 0$, and that

$$h'_-(x,0) = \int_{-\infty}^{0} q(t)h_-(t,0)dt \geq 0.$$

But since $h_- \to 1$ at $-\infty$ and $h'_- \geq 0$ we see that $h_-(x,0) \geq 1$ for all x. So $f_-(x,0) = h_-(x,0) \geq 1$ in R and f_- is not a multiple of f_+. □

Example 2. There are potentials q in $P(1,N)$ for $N \geq 1$ such that $W(0) = 0$. One such function is given

$$q(x) = \begin{cases} 0 & \text{if } x < -2 \\ b & \text{if } -2 < x < -1 \\ -(\pi/2)^2 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 < x \end{cases}$$

where b is determined uniquely by the conditions

$$0 < b < 1$$

and $b^2 = x$ where x is the positive solution of

$$e^{2x} = \frac{x + (\pi/2)^2}{x - (\pi/2)^2}.$$

This may be verified by computing $f_+(x,0)$ and $f_-(x,0)$ explicitly. □

Chapter II: The Inverse Scattering Theory

Section 4. Statement and clarification of the inverse problem. In Chapter I we derived a list of necessary conditions (D.1-D.7) satisfied by the scattering data of a potential in the class $P(c,2)$, $c > 0$. The eventual goal of this chapter is to show that the conditions (D.1-D.7) are also sufficient to characterize the scattering data of a potential in $P(c,2)$, i.e. to show that if candidate set $\mathcal{S}$ of scattering data has properties (D.1-D.7), then there exists a unique potential v in $P(c,2)$ such that the scattering data of v is exactly $\mathcal{S}$.

On the way to this goal we will prove a slightly stronger though more technical result by giving weaker sufficient conditions for candidate scattering data to be the scattering data of a potential in $P(c,1)$. The point of the greater technicality is to analyze carefully the method of Faddeev and to elucidate as well as possible where the Buslaev-Fomin paper went wrong.

To begin we consider the following candidate scattering data:

$$\begin{bmatrix} T_-(k) & R_-(k) \\ R_+(k) & T_+(k) \end{bmatrix}, \quad \{i\kappa_j : j \in J\}, \quad \{c_{+j} : j \in J\}, \quad \{\mu_j : j \in J\} \tag{4.1}$$

where R_+ and R_- are defined for $0 \neq k \in \mathbf{R}$, T_+ and T_- are defined for $\operatorname{Im} k \geq 0$ and $k \neq 0$, J is a finite index set with $0 \leq \mathcal{J} = \#J < \infty$, and

$$\kappa_j > 0, \quad c_{+j} > 0, \quad \mu_j \neq 0 \qquad \text{for all } j \in J.$$

If J is not empty we make the convention that $J = \{1,2,\ldots,\mathcal{J}\}$ and that $0 < \kappa_1 < \kappa_2 < \ldots < \kappa_{\mathcal{J}}$. We use the function $\ell = \ell(k) = \sqrt{k^2 - c^2}$ as introduced in Chapter I, and make the definitions

$$\lambda_j = \sqrt{\kappa_j^2 + c^2} \qquad \text{for } j \in J$$

$$c_{-j} = c_{+j}/(\mu_j)^2 \qquad \text{for } j \in J.$$

Note that $i\lambda_j = \ell(i\kappa_j)$. One final notational convention:

$$\tilde{R}_+(\ell) = R_+(k(\ell)) \qquad \text{for } \ell \in \mathbf{R}$$

$$\tilde{T}_+(\ell) = T_+(k(\ell)) \qquad \text{for } \operatorname{Im} \ell \geq 0 \text{ but } \ell \notin \{i\lambda : 0 \leq \lambda \leq c\}$$

where $k = k(\ell)$ is the inverse function for $\ell = \ell(k)$.

Initially we assume that this data satisfies the conditions (X.1-X.6) developed below:

(X.1)

(i) $\tilde{R}_+(\ell)$ is a complex-valued function of real ℓ

(ii) $\tilde{R}_+(\ell)$ is in $L^2(\mathbf{R})$

(iii) $\tilde{R}_+(\ell)$ is continuous on $\mathbf{R} \sim \{0\}$

(iv) $\tilde{R}_+(-\ell) = \tilde{R}_+^*(\ell)$ for $0 \neq \ell \in \mathbf{R}$

(v) $\tilde{R}_+(\ell) = O(\ell^{-1})$ as $\ell \to \pm\infty$.

Let $F_+(x)$ be the inverse Fourier transform of $\tilde{R}_+(\ell)$ defined by

$$F_+(x) = \pi^{-1} \int_{\ell=-\infty}^{\infty} \tilde{R}_+(\ell) e^{2i\ell x} d\ell.$$

By (X.1 iv) $F_+(x)$ is real valued, and we have

$$\tilde{R}_+(\ell) = \int_{x=-\infty}^{\infty} F_+(x)e^{-2i\ell x}dx$$

in L^2 and in Cesaro mean. The second condition we impose is the Buslaev-Fomin condition, which is weaker than (D.6).

(X.2) F_+ is absolutely continuous. For all finite X

$$\int_X^{\infty} |F'_+(x)|(1 + |x|)dx < \infty.$$

Similarly for R_- we assume

(X.3)
- (i) $R_-(k)$ is a complex-valued function on $\mathbf{R} \sim \{0\}$
- (ii) $R_-(k) \in L^2(\mathbf{R})$
- (iii) $R_-(k)$ is continuous on $\mathbf{R} \sim \{0\}$
- (iv) $R_-(-k) = R_-^*(k)$ for $O \neq k \in \mathbf{R}$
- (v) $R_-(k) = O(k^{-1})$ as $k \to \pm\infty$.

Setting

$$F_-(x) = \pi^{-1} \int_{-\infty}^{\infty} R_-(k)e^{-2ikx}dx$$

we find that $F_-(x)$ is real valued and that

$$R_-(k) = \int_{-\infty}^{\infty} F_-(x)e^{+2ikx}dx.$$

For F_- we assume

(X.4) F_- is absolutely continuous. Further, for all finite X,

$$\int_{-\infty}^{X} |F'_-(x)|(1 + |x|)dx < \infty.$$

Finally we assume

(X.5)
- (i) $T_+(k)$ is meromorphic in $\{k: \text{Im } k > 0\}$
- (ii) The set $\{i\kappa_j : j \in J\}$ is the set of poles of T_+; these poles are all simple
- (iii) $T_+(-k) = T_+^*(k)$ for $0 \neq k \in \mathbf{R}$
- (iv) $\int_{-c}^{c} |kT_+(k)/\ell|^2 dk < \infty$
- (v) $T_+(k) \neq 0$ for $k \in \mathbf{R}$ with $|k| > c$
- (vi) $\tilde{T}_+\{\ell\}$ is continuous everywhere in $\{\ell: \text{Im } \ell \geq 0\}$ except at the poles $i\lambda_j$ and on the segment $\{i\lambda: 0 \leq \lambda \leq c\}$
- (vii) $\tilde{T}_+(\ell) = 1 + O(\ell^{-1})$ as $|\ell| \to \infty$ uniformly in $\text{Im } \ell \geq 0$.

(X.6)
(i) $T_-(k)$ is meromorphic in $\{k: \operatorname{Im} k > 0\}$
(ii) $\{i\kappa_j : j \in J\}$ is the set of poles of T_-; these poles are simple
(iii) $T_-(-k) = T_-^*(k)$ for $0 \neq k \in \mathbf{R}$
(iv) $\int_0^c |T_-(k)|^2 dk < \infty$
(v) $T_-(k) \neq 0$ for $0 \neq k \in \operatorname{Dom}(T)$
(vi) $T_-(k)$ is continuous in $\{k: \operatorname{Im} k \geq 0\}$ except at 0 and at the poles $i\kappa_j$
(vii) $T_-(k) = 1 + O(k^{-1})$ as $|k| \to 0$ $\operatorname{Im} k \geq 0$.

It is easy to verify that the conditions (D.1-D.7) imply the conditions (X.1-X.6). The conditions (X.1-X.6) are not sufficient to insure that the data (4.1) are scattering data of a v in $P(c,1)$. Note that (X.1-X.6) include no conditions relating R_+, T_+ to R_-, T_- except for parts (ii) of (X.5) and (X.6). But we work with them alone for the first part of the inverse scattering construction. Chapter I says that if the data (4.1) arise from a potential v of class $P(c,1)$, then v is uniquely determined as

(4.2) $$v(x) = -\partial_x B_+(x,0) + c^2 = \partial_x B_-(x,0) \qquad \text{almost everywhere}$$

where $B_+(x,y)$ and $B_-(x,y)$ satisfy the Marchenko equations (1.31) and (1.32).

Now we repeat the definitions (familiar from Chapter I) of G_+, H_+, Ω_+, G_-, and Ω_-

$$G_+(x) = 2 \sum_J c_{+j} e^{-2\lambda_j x}$$

$$H_+(x) = \pi^{-1} \int_0^c |T_-(k)|^2 e^{-2\sqrt{c^2-\kappa^2}x}\, dk$$

$$\Omega_+(x) = F_+(x) + G_+(x) + H_+(x)$$

$$G_-(x) = 2 \sum_J c_{-j} e^{2\kappa_j x}$$

$$\Omega_-(x) = F_-(x) + G_-(x).$$

We now consider the Marchenko equations

(4.3) $$B_-(x,y) + \Omega_-(x+y) + \int_{-\infty}^0 B_-(x,z)\Omega_-(x+y+z)dz = 0$$

(4.4) $$B_+(x,y) + \Omega_+(x+y) + \int_0^\infty B_+(x,z)\Omega_+(x+y+z)dz = 0$$

as integral equations for unknowns $B_-(x,\cdot) \in L^1((-\infty,0])$ and $B_+(x,\cdot) \in L^1([0,\infty))$ for each x in $\mathbf{R}$.

By (X.2), (X.4) and (X.6iv) one checks that

$$\int_{-\infty}^X |\Omega_-'(x)|(1+|x|)dx < \infty \quad \text{and} \quad \int_X^\infty |\Omega_+'(x)|(1+|x|)dx < \infty$$

for all finite X. Thus by [1] we know that the solutions $B_\pm$ of (4.3) and (4.4) exist and are absolutely continuous in each variable separately. It will be convenient to extend the domains of $B_\pm$ by setting

$$B_-(x,y) = 0 \quad \text{if } y > 0; \qquad B_+(x,y) = 0 \quad \text{if } y < 0.$$

Set

$$v_+(x) = -\partial_x B_+(x,0) + c^2 \tag{4.5}$$

$$v_-(x) = -\partial_x B_+(x,0) \tag{4.6}$$

The burden of this chapter is to see when $v_+ = v_-$ and when v_+ is a potential in $P(c,N)$ having the objects (4.1) as scattering data.

Introduce the following functions:

$$h_+(x,\ell) = 1 + \int_0^\infty B_+(x,y)e^{2i\ell y}dy \qquad \text{for Im } \ell \geq 0$$

$$f_+(x,k) = e^{i\ell x}h_+(x,\ell(k)) \qquad \text{for Im } k \geq 0$$

$$g_+(x,\ell) = \frac{h_+(x,-\ell) + \tilde{R}_+(\ell)e^{2i\ell x}h_+(x,\ell)}{\tilde{T}_+(\ell)} \qquad \text{for } 0 \neq \ell \in \mathbf{R}$$

and

$$h_-(x,k) = 1 + \int_{-\infty}^0 B_-(x,z)e^{-2ikz}dz \qquad \text{for Im } k \geq 0$$

$$f_-(x,k) = e^{-ikx}h_-(x,k) \qquad \text{for Im } k \geq 0$$

$$g_-(x,k) = \frac{h_-(x,-k) + R_-(k)e^{-2ikx}h_-(x,k)}{T_-(k)} \qquad \text{for } 0 \neq k \in \mathbf{R}.$$

By [1] we know that $B_\pm(x,\cdot)$ are in L^1 for each x so $f_+(x,k) \sim e^{i\ell x}$ and $f_-(x,k) \sim e^{-ikx}$ as $x \to +\infty$ and $-\infty$ respectively. Agranovich and Marchenko [1, p. 117] also show why f_+ solves the Schrödinger equation

$$-y'' + v_+(x)y = k^2y \qquad \text{in Im } k \geq 0 \tag{4.7+}$$

and why f_- solves

$$-y'' + v_-(x)y = k^2y \qquad \text{in Im } k \geq 0. \tag{4.7−}$$

Lemma 4.1. *In order to conclude that $v_+(x) = v_-(x)$ almost everywhere on $\mathbf{R}$ it suffices to show that*

$$T_-(k)f_+(x,k) = f_-^*(x,k) + R_-(k)f_-(x,k) \tag{4.8}$$

for all x in $\mathbf{R}$ and all real k with $|k| > c$.

Proof. Fix $k \in \mathbf{R}$ with $|k| > c$. Relation (4.8) says that $f_+(x,k)$ is a linear combination of $f_-^*(x,k)$ and $f_-(x,k)$. But since f_-^* solves (4.7−) as well as f_-

itself, we see that f_+ solves both (4.7−) and (4.7+). Subtraction yields

$$\{v_+(x) - v_-(x)\}f_+(x,k) = 0, \qquad \text{a.e.}$$

Thus $v_+(x) = v_-(x)$ whenever $f_+(x,k) \neq 0$.

It remains to show that $f_+(x,k)$ cannot vanish for $x \in \mathbf{R}$. Suppose there were an x_0 where $f_+(x_0,k) = 0$. But then $f_+^*(x_0,k) = 0$, too. But since $|k| > c$ we have $\ell(k) \in \mathbf{R}$. So f_+ and f_+^* are independent solutions of (4.7+)—the independence follows from their behavior as $x \to +\infty$—and independent solutions of (4.7+) cannot vanish at the same x. Therefore $v_+ = v_-$ on $\mathbf{R}$. □

The rest of this chapter is devoted to the following tasks:

Step 1. Show that

$$T_-(k)f_+(x,k) = f_-^*(x,k) + R_-(k)f_-(x,k) \tag{4.8+}$$

$$T_+(k)f_-(x,k) = f_+^*(x,k) + R_+(k)f_+(x,k) \tag{4.8−}$$

for $x \in \mathbf{R}$ and $k \in R$, $|k| > c$, whence $v_+(x) = v_-(x)$ on $\mathbf{R}$.

Step 2. Let v denote the function $v(x) = v_+(x) = v_-(x)$. Show that $v \in P(c,1)$.

Step 3. Show that the objects (4.1) are the scattering data of v.

To accomplish this program we will need to strengthen the hypothesis on the candidate scattering data. Step 1 is done in Section 5 using two sizable theorems. Steps 2 and 3 are done in Section 6.

Section 5. To show that $v_+ = v_-$.

Lemma 5.1. *Assume that (X.1-X.6) hold. Then $g_+(x,\ell)$ extends analytically to $\{\ell : \operatorname{Im} \ell > 0\} \sim \{i\lambda : 0 < \lambda \leq c\}$.*

Proof. By definition

$$\tilde{T}_+(\ell)g_+(x,\ell) = h_+(x,-\ell) + \tilde{R}_+(\ell)e^{2i\ell x}h_+(x,\ell) \qquad \text{for } 0 \neq \ell \in \mathbf{R}.$$

Since

$$h_+(x,\ell) = 1 + \int_0^\infty B_+(x,z)e^{2i\ell z}dz$$

and

$$\tilde{R}_+(\ell) = \int_{-\infty}^{\infty} F_+(x)e^{-2i\ell x}dx$$

one finds that

$$\tilde{T}_+(\ell)g_+(x,\ell) - 1 = \int_0^\infty \mathcal{B}_+(x,y)e^{-2i\ell y}dy$$

where

$$\mathcal{B}_+(x,y) = B_+(x,y) + F_+(x+y) + \int_0^\infty B_+(x,z)F_+(x+y+z)dz.$$

For each fixed x, consider $\mathcal{B}_+$ as a function of y. Since $B_+(x,y)$ is bounded and L^1, it is L^2. F_+ is in L^2 because it is the transform of the L^2 function $\tilde{R}_+$. The last term in $\mathcal{B}_+$ is essentially the convolution of an L^1 function with an L^2 function, so it is in L^2. Thus $\mathcal{B}_+(x,y)$ is L^2 as a function of y. Now for $y > 0$ we can apply the Marchenko equation to see that

$$\mathcal{B}_+(x,y) = -G_+(x+y) - \int_0^\infty B_+(x,z)G_+(x+y+z)dz$$
$$- H_+(x,y) - \int_0^\infty B_+(x,z)H_+(x+y+z)dz.$$

Computing for $0 \neq \ell \in \mathbf{R}$ one gets

$$\int_{y=0}^\infty \left[-G_+(x+y) - \int_0^\infty B_+(x,z)G_+(x+y+z)dz\right]e^{-2i\ell y}dy$$
$$= \frac{i}{2\pi}\sum_J c_{+j}e^{-2\lambda_j x}h_+(x,i\lambda_j)/(\ell - \lambda_j)$$

and

$$\int_{y=0}^\infty \left[-H_+(x+y) - \int_0^\infty B_+(x,z)H_+(x+y+z)dz\right]e^{-2i\ell y}dy$$
$$= \frac{i}{2\pi}\int_{\sigma=0}^c |T_-(\sigma)|^2 e^{-\sqrt{c^2-\sigma^2}x}\,\frac{1}{\ell - i\sqrt{c^2-\sigma^2}}\,h_+(x,i\sqrt{c^2-\sigma^2})d\sigma.$$

Thus, still for $0 \neq \ell \in \mathbf{R}$,

$$\text{(5.1)}\quad T_+(\ell)g_+(x,\ell) - 1 = \int_{y=-\infty}^0 \mathcal{B}_+(x,y)e^{-2i\ell y}dy$$
$$+ i\sum c_{+j}e^{-2\lambda_j x}h_+(x,i\lambda_j)/(\ell - i\lambda j)$$
$$+ \frac{i}{2\pi}\int_{\sigma=0}^c |T_-(\sigma)|^2 e^{-\sqrt{c^2-\sigma^2}x}\,\frac{h_+(x,i\sqrt{c^2-\sigma^2})d\sigma}{\ell - i\sqrt{c^2-\sigma^2}}.$$

Since $\mathcal{B}_+(x,y)$ is L^2 on $-\infty < y < 0$, the first term extends to $\operatorname{Im}\ell > 0$ as an analytic function of class H^{2+}. The second term is meromorphic in $\operatorname{Im}\ell > 0$ with poles at the $i\lambda_j$. The third term is analytic in $\{\ell : \operatorname{Im}\ell > 0\} \sim \{i\lambda : 0 < \lambda \leq c\}$. But, since

$$g_+(x,\ell) = \frac{\{\tilde{T}_+(\ell)g_+(x,\ell) - 1\} + 1}{\tilde{T}_+(\ell)},$$

the poles of $\tilde{T}_+(\ell)$ cancel the poles of $\tilde{T}_+(\ell)g_+(x,\ell) - 1$, and $g_+(x,\ell)$ extends to $\{\ell : \text{Im}\ \ell > 0\} \sim \{i\lambda : 0 < \lambda \leq c\}$ as an analytic function. □

Theorem 5.2. *Assume that the candidate scattering data has properties (X.1-X.6). Suppose (h,g) is a pair of functions such that*

(K.0)
$$h = h(x,\ell) \text{ is defined for } x \text{ in } \mathbf{R} \quad \text{and} \quad \text{Im}\ \ell \geq 0,\ \ell \neq 0, \text{ and}$$
$$g = g(x,\ell) \text{ is defined for } x \text{ in } \mathbf{R} \quad \text{and} \quad \text{Im}\ \ell \geq 0,\ \ell \in \{i\lambda : 0 \leq \lambda \leq c\}.$$

Then in order to show that $(h,g) = (h_+,g_+)$ it suffices to verify that the following conditions are met for each real x:

(K.1) *$h(x,\ell) - 1 \in H^{2+}$; $h(x,\ell)$ is continuous on $\{\ell : \text{Im}\ \ell \geq 0\} \sim \{0\}$.*

(K.2.1) *$g(x,\ell)$ is analytic in $\{\ell : \text{Im}\ \ell > 0\} \sim \{i\lambda : 0 < \lambda \leq c\}$ and continuous on $\{\ell : \text{Im}\ \ell \geq 0\} \sim \{i\lambda : 0 \leq \lambda \leq c\}$.*

(K.2.2) *$\tilde{T}_+(\ell)g(x,\ell) - 1 \in L^2(\mathbf{R})$.*

(K.3) *For $0 \neq \ell \in \mathbf{R}$, $h(x,-\ell) = h^*(x,\ell)$ and $g(x,-\ell) = g^*(x,\ell)$.*

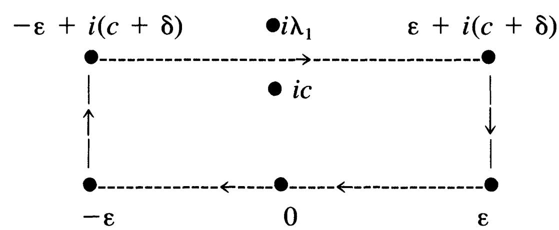
(K.4) *$g(x,\ell) = 1 + O(\ell^{-1})$ as $|\ell| \to \infty$ uniformly in $\text{Im}\ \ell \geq 0$.*

(K.5.1) *For $0 \neq \ell \in \mathbf{R}$, $\tilde{T}_+(\ell)g(x,\ell) = h(x,-\ell) + \tilde{R}_+(\ell)e^{2i\ell x}h(x,\ell)$.*

(K.5.2) *If $\varepsilon > 0$, $\delta > 0$, and $c + \delta < \inf\{\lambda_j : j \in J\}$, then*

$$\int_{L(\varepsilon,\delta)} e^{2iy\ell}\{\tilde{T}_+(\ell)g(x,\ell) - 1\}d\ell$$
$$= \int_0^c |T_-(k)|^2 e^{-2(x+y)\sqrt{c^2-k^2}}\ h(x, i\sqrt{c^2 - k^2})dk$$

where $L(\varepsilon,\delta)$ is the rectangular path $-\varepsilon$ to $-\varepsilon + i(c + \delta)$ to $\varepsilon + i(c + \delta)$ to ε and back to $-\varepsilon$ as shown below.



(K.5.3) $\text{Res}(\tilde{T}_+(\ell)g_+(x,\ell), i\lambda_j) = ic_{+j}e^{-2\lambda_j x}h(x, i\lambda_j)$ *for j in J.*

Proof. Suppose that (h,g) does satisfy (K.0-K.5). Fix x. By (K.1) there is a function $A(x,y)$ such that

$$A(x,\cdot) \in L^2(\mathbf{R}); \qquad A(x,y) = 0 \quad \text{for } y < 0;$$

$$h(x,\ell) - 1 = \int_{y=0}^{\infty} A(x,y)e^{2i\ell y}dy. \tag{5.2}$$

By (K.3), A is real valued. Using (5.1) and the relation

$$\tilde{R}_+(\ell) = \int_{-\infty}^{\infty} F_+(x)e^{-2i\ell x}dx$$

in the equation (K.5.1) we get

$$\tilde{T}_+(\ell)g(x,\ell) - 1 = \int_{-\infty}^{\infty} [A(x,y) + F_+(x+y) + \int_0^{\infty} F_+(x+y+z)A(x,z)dz]e^{-2i\ell y}dy.$$

Inverting the transform gives

$$A(x,y) + F_+(x+y) + \int_0^{\infty} F_+(x+y+z)A(x,z)dz = \pi^{-1}\int_{-\infty}^{\infty} \{\tilde{T}_+(\ell)g(x,\ell) - 1\}e^{2i\ell y}d\ell \tag{5.3}$$

for $y \in \mathbf{R}$. We set $y = y' > 0$ and look at the integrand

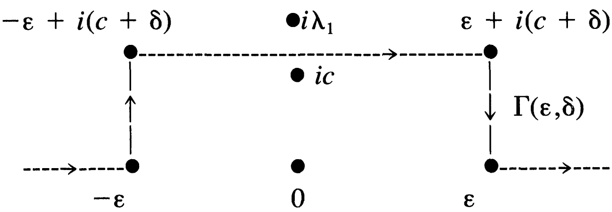
$$\mathcal{T}(\ell) \equiv \{\tilde{T}_+(\ell)g(x,\ell) - 1\}e^{2i\ell y'}$$

as a function of $\ell = \ell_1 + i\ell_2$. Clearly $\mathcal{T}$ is meromorphic in $\{\ell : \operatorname{Im} \ell > 0\} \sim \{i\lambda : 0 < \lambda \leq c\}$ with poles at the points $i\lambda_j$. $\mathcal{T}$ extends continuously down to $\{\ell : \operatorname{Im} \ell = 0\} \sim \{0\}$, and $\mathcal{T}$ is in L^2 on $\mathbf{R}$. Finally, $\mathcal{T}(\ell) = O(\ell^{-1})$ as $|\ell| \to \infty$ with $\operatorname{Im} \ell \geq 0$ and $\mathcal{T}(\ell)$ decays exponentially as $\ell_2 \to +\infty$ uniformly for ℓ_1 in any compact interval. These results depend on the assumptions $\tilde{T}_+ - 1 = O(\ell^{-1})$ and $g - 1 = O(\ell^{-1})$.

Consider first the integral

$$\pi^{-1}\int_{\Gamma(\varepsilon,\delta)} \{\tilde{T}_+(\ell)g(x,\ell) - 1\}e^{2i\ell y'}d\ell \equiv \mathcal{J}_1(x,y')$$

where $\Gamma(\varepsilon,\delta)$ is the polygonal curve from $-\infty$ to $-\varepsilon$, to $-\varepsilon + i(c+\delta)$ to $\varepsilon + i(c+\delta)$ to ε to $+\infty$ as shown below



Because of the results of the previous paragraph we can close the contour in the upper half-plane, to get

$$\mathcal{J}_1(x,y') = 2i \sum_J \operatorname{Res}(\tilde{T}_+(\ell)g(x,\ell)\}e^{2i\ell y'},i\lambda_j) = 2i \sum_J e^{-2\lambda_j y'} \operatorname{Res}(\tilde{T}_+ g,i\lambda).$$

Employing (K.5.3) and (5.2) we now get

$$\begin{aligned}\mathcal{J}_1(x,y') &= 2i \sum_J e^{-2\lambda_j y'} ic_{+j} e^{-2\lambda_j x} h(x,i\lambda_j) \\ &= -2i \sum_J e^{-2\lambda_j(x+y')} ic_{+j}\left[1 + \int_0^\infty A(x,z)e^{-2\lambda z}dz\right].\end{aligned}$$

Recalling the definition of G_+, we see

$$\pi^{-1}\mathcal{J}_1(x,y') = -G_+(x,y') + \int_0^\infty A(x,z)G_+(x + y' + z)dz. \tag{5.4}$$

Now note that $\Gamma(\varepsilon,\delta)$ is the sum of $\mathbf{R}$ and $L(\varepsilon,\delta)$. So, for $y' > 0$,

$$\begin{aligned}\pi^{-1}\int_{\mathbf{R}} \{\tilde{T}_+(\ell)g(x,\ell) - 1\}e^{2i\ell y'}d\ell &= \pi^{-1}\int_{\mathbf{R}} \mathcal{T}(\ell)d\ell \\ &= \pi^{-1}\int_{\Gamma(\varepsilon,\delta)} \mathcal{T}(\ell)d\ell - \pi^{-1}\int_{L(\varepsilon,\delta)} \mathcal{T}(\ell)d\ell \\ &= \mathcal{J}_1 - \pi^{-1}\int_{L(\varepsilon,\delta)} \{\tilde{T}_+(\ell)g(x,\ell) - 1\}e^{2i\ell y'}d\ell.\end{aligned} \tag{5.5}$$

By (K.5.3) (5.2), and the definition of $H_+(x)$

$$\begin{aligned}&\pi^{-1}\int_{L(\varepsilon,\delta)} \{\tilde{T}_+(\ell)g(x,\ell) - 1\}e^{2i\ell y'}d\ell \\ &\quad = \pi^{-1}\int_0^c |T_-(k)|^2 e^{-2(x+y')\sqrt{c^2-k^2}}\, h(x,i\sqrt{c^2 - k^2})dk \\ &\quad = \pi^{-1}\int_0^c |T_-(k)|^2 e^{-2(x+y')\sqrt{c^2-k^2}}\left[1 + \int_0^\infty A(x,z)e^{-2z\sqrt{c^2-k^2}}dz\right]dk \\ &\quad = H_+(x + y') + \int_0^\infty A(x,z)H_+(x + y' + z)dz.\end{aligned} \tag{5.6}$$

Combining (5.3) on the left of (5.5) with (5.4) and (5.6) on the right we obtain, for $y' > 0$,

$$A(x,y') + F_+(x+y') + \int_0^\infty A(x,z)F_+(x+y'+z)dz$$
$$= -G_+(x+y') - \int_0^\infty A(x,z)G_+(x+y'+z)dz$$
$$- H_+(x+y') - \int_0^\infty A(x,z)H_+(x+y'+z)dz$$

or

$$A(x,y') + \Omega_+(x+y') + \int_0^\infty A(x,z)\Omega_+(x+y'+z)dz = 0.$$

But this says that $A(x,\cdot)$ solves the Marchenko equation (4.4). Since (4.4) has a unique solution, $A(x,\cdot) = B_+(x,\cdot)$ and

$$h(x,\ell) = 1 + \int_0^\infty B(x,y)e^{2i\ell z}dz = h_+(x,\ell) \qquad \text{for } \operatorname{Im} \ell \geq 0.$$

But then by (K.5.1) and the definition of g_+

$$g(x,\ell) = \frac{h_+(x,-\ell) + \tilde{R}_+(\ell)e^{2i\ell x}h_+(x,\ell)}{\tilde{T}_+(\ell)} = g_+(x,\ell) \qquad \text{for } 0 \neq \ell \in \mathbf{R}.$$

By Lemma 5.1 we know that g_+ extends to be analytic in $\{\ell : \operatorname{Im} \ell > 0\} \sim \{i\lambda : 0 \leq \lambda \leq c\}$. Since $g(x,\ell)$ and $g_+(x,\ell)$ agree on $\mathbf{R} \sim \{0\}$, we conclude that $g = g_+$ in $\{\ell : \operatorname{Im} \ell \geq 0\} \sim \{i\lambda : 0 \leq \lambda \leq c\}$ [10, p. 128 ff.]. □

To complete Step 1 we will verify

(4.8+) $$T_-(k)f_+(x,k) = f_-^*(x,k) + R_-(k)f_-(x,k)$$

(4.8−) $$T_+(k)f_-(x,k) = f_+^*(x,k) + R_+(k)f_+(x,k)$$

for all real x and all real k with $|k| > c$. From the definitions of f_+, f_-, g_+, g_- in terms of h_+ and h_- it follows that (4.8±) are equivalent to

(5.7+) $$h_+(x,\ell) = e^{-i\ell x}e^{ik(\ell)x}g_-(x,k(\ell))$$

(5.7−) $$g_+(x,\ell) = e^{i\ell x}e^{-ik(\ell)x}h_-(x,k(\ell))$$

for all real ℓ with $\ell \neq 0$. We now define

$$h_0(x,\ell) = e^{-i\ell x}e^{ik(\ell)x}g_-(x,k(\ell)) = \text{right side of (5.7+)}$$

and

$$g_0(x,\ell) = e^{i\ell x}e^{-ik(\ell)x}h_-(x,k(\ell)) = \text{right side of (5.7−)}.$$

It now remains to show that (h_0,g_0) satisfies the conditions (K.0-K.5) of Theorem 5.2. To do this we use the following list of hypotheses (Y.1-Y.7) on the candidate scattering data, which is stronger than (X.1-X.6) but still weaker than (D.1-D.7):

(Y.1) $T_\pm(-k) = T^*_\pm(k)$ and $R_\pm(-k) = R^*_\pm(k)$ for $0 \neq k \in \mathbf{R}$.

(Y.2)
(i) $R_-(k)$ is continuous on $\mathbf{R} \sim \{0\}$; $R_+(k)$ is continuous on $\mathbf{R} \sim [-c,c]$.
(ii) $R_-(k)$ and $\tilde{R}_+(\ell)$ are in $L^2(\mathbf{R})$.
(iii) $T_\pm(k)$ are meromorphic on $\{k: \mathrm{Im}\ k > 0\}$ and continuous on $\{k: 0 \leq \mathrm{Im}\ k < \underline{\kappa}\} \sim \{0\}$ where $\underline{\kappa} = \inf \kappa_j$.
(iv) T_- extends continuously to $k = 0$.

(Y.3) The poles of T_+ are the same as the poles of T_-, namely the points $i\kappa_j$ for $j \in J$. These poles are all simple and

$$\mathrm{Res}(T_-, i\kappa_j) = \mathrm{sgn}(\mu_j) i \sqrt{c_{+j} c_{-j}}.$$

(Y.4)
(i) If $\mathrm{Im}\ k \geq 0$ and $k \notin \{i\kappa_j : j \in J\}$ then $kT_+(k) = \ell T_-(k)$.
(ii) If $k \in \mathbf{R}$ and $|k| > c$, then

$$1 = \frac{k}{\ell}|T_+|^2 + |R_+|^2 = \frac{\ell}{k}|T_-|^2 + |R_-|^2 \qquad 0 = \ell T_- R^*_- + kT^*_+ R_+.$$

(iii) If $k \in \mathbf{R}$ and $0 < |k| \leq c$, then $R_-(k) = T_-(k)/T^*_-(k)$.
(iv) If $\mathrm{Im}\ k \geq 0$, $k \neq 0$, and $k \notin \{i\kappa_j : j \in J\}$ then $T_-(k) \neq 0$.

(Y.5) $T_\pm(k) = 1 + O(k^{-1})$ as $|k| \to \infty$ uniformly in $\mathrm{Im}\ k \geq 0$.
$R_\pm(k) = O(k^{-1})$ as $k \to \pm\infty$, $k \in \mathbf{R}$.

(Y.6) The functions F_+ and F_- are absolutely continuous and L^2 on $\mathbf{R}$. Further

$$\int_X^\infty |F'_+(x)|(1 + |x|)dx < \infty \qquad \text{for all finite } X$$

and

$$\int_{-\infty}^X |F'_-(x)|(1 + |x|)dx < \infty \qquad \text{for all finite } X.$$

Condition (Y.7) is stated after the proof of Theorem 5.3. It is straightforward to verify that the conditions (D.1-D.7) imply (Y.1-Y.6) and that (Y.1-Y.6) imply (X.1-X.6). Note that (Y.3) and (Y.4) connect R_+, T_+ strongly to R_-, T_-.

Theorem 5.3. *Suppose the candidate scattering data satisfy properties (Y.1-Y.6). If $h_0(x,\ell) - 1$ is in H^{2+} as a function of ℓ, then the pair (h_0, g_0) has properties (K.0-K.5), whence $(h_0, g_0) = (h_+, g_+)$ and the relations (4.8±) hold for $x \in \mathbf{R}$, $k \in \mathbf{R} \sim [-c,c]$.*

Proof. Recall that for $0 \neq \ell \in \mathbf{R}$

$$h_0(x,\ell) = e^{-i\ell x}e^{ikx}\left\{\frac{h_-(x,-k) + R_-(k)e^{-2ikx}h_-(x,k)}{T_-(k)}\right\}$$

and

$$g_0(x,\ell) = e^{i\ell x}e^{-ikx}h_-(x,k)$$

where $k = k(\ell)$. For ℓ real, $k(\ell) = \operatorname{sgn}(\ell)\sqrt{\ell^2 + c^2}$ is continuous except for a jump at $\ell = 0$. It follows that $h_0(x,\ell)$ is continuous on $\mathbf{R} \sim \{0\}$. Since $h_0(x,\ell) - 1 \in H^{2+}$ by hypothesis, the Poisson formula tells us that $h_0(x,\ell)$ is continuous in $\{\ell : \operatorname{Im} \ell \geq 0\} \sim \{0\}$. Thus (h_0, g_0) satisfies (K.0) and (K.1).

We know that $h_-(x,k)$ is analytic for $\operatorname{Im} k > 0$ and continuous for $\operatorname{Im} k \geq 0$. We also know that $\ell = \ell(k)$ is analytic for $\operatorname{Im} \ell > 0$ where $\ell \notin \{i\lambda : 0 < \lambda \leq c\}$ and continuous down to the real axis, except at 0. Thus g_0 satisfies (K.2.1).

Consider

$$\mathcal{T}(\ell) = \tilde{T}_+(\ell)g_0(x,\ell) - 1 = T_+(k(\ell))e^{i\ell x}e^{-ikx}h_-(x,k) - 1$$

for $k = k(\ell)$ and $\ell \in \mathbf{R}$. As ℓ approaches 0 from left (right), k approaches $-c$ from the left ($+c$ from the right). Since $T_+(k)$ and $h_-(x,k)$ are continuous on $\mathbf{R} \sim \{0\}$, it follows that $\mathcal{T}(\ell)$ is continuous on $\mathbf{R}$ except for a finite jump at $\ell = 0$.

We know that

$$k = k(\ell) = \ell + O(\ell^{-1}) \quad \text{and} \quad h_-(x,k(\ell)) = 1 + O(k^{-1}) = 1 + O(\ell^{-1})$$

as $\ell \to \pm\infty$. Further,

$$e^{i\ell x}e^{-ikx} = e^{i(\ell-k)x} = 1 + O(\ell^{-1})$$

as $\ell \to \pm\infty$. Thus it follows that

$$\mathcal{T}(\ell) = \{1 + O(\ell^{-1})\}\{1 + O(\ell^{-1})\}\{1 + O(\ell^{-1})\} - 1 = O(\ell^{-1})$$

as $\ell \to \pm\infty$, whence $\mathcal{T}(\ell) \in L^2(\mathbf{R})$ and (K.2.2) holds.

Verification of (K.3) is routine.

For (K.4), note that

$$g_+(x,\ell) = e^{i(\ell-k)x}\left\{1 + \int_{-\infty}^{0} B_-(x,y)e^{-2iky}dy\right\}$$

$$= \{1 + O(\ell^{-1})\}\left\{1 - \frac{1}{2ik}\int_{-\infty}^{0} B_-(x,y)\frac{d}{dy}[e^{-2iky}]dy\right\}.$$

After integration by parts we get

$$g_+(x,\ell) = \{1 + O(\ell^{-1})\}\{1 + O(k^{-1})\} = 1 + O(\ell^{-1}).$$

It remains to verify (K.5.1-K.5.3). For $0 \neq \ell \in \mathbf{R}$, i.e. for $k \in \mathbf{R}$ with $|k| > c$, the definitions of h_0 and g_- yield

$$h_0(x,-\ell) + \tilde{R}_+(\ell)e^{2i\ell x}h_0(x,\ell) = e^{i(\ell-k)x}\{g_-(x,-k) + R_+(k)e^{2ikx}g_-(x,k)\}$$

$$= e^{i(\ell-k)x}h_-(x,k)\left\{\frac{1}{T_-(-k)} + \frac{R_+(k)R_-(k)}{T_-(k)}\right\}$$

$$+ e^{i(k-\ell)x}h_-(x,-k)\left\{\frac{R_-(-k)}{T_-(-k)} + \frac{R_+(k)}{T_-(k)}\right\}.$$

But by (Y.3) with $k \in \mathbf{R}$, $|k| > c$ and (Y.1)

$$\frac{1}{T_-(-k)} + \frac{R_+(k)R_-(k)}{T_-(k)} = T_+(k)$$

and

$$\frac{R_-(-k)}{T_-(-k)} + \frac{R_+(k)}{T_-(k)} = 0.$$

It follows that

$$h_0(x,-\ell) + \tilde{R}_+(\ell)e^{2i\ell x}h_0(x,\ell) = \tilde{T}_+(k)h_-(x,k)e^{i(\ell-k)x} = \tilde{T}_+(\ell)g_0(x,\ell)$$

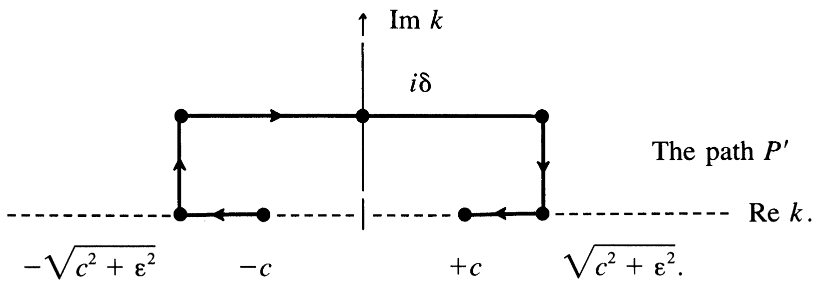
for $0 \neq \ell \in \mathbf{R}$, which is (K.5.1).

We next turn to (K.5.2). Pick $\varepsilon > 0$, and $\delta > 0$ such that $c + \delta < \inf\{\lambda_j : j \in J\}$. Consider the box $L = L(\varepsilon,\delta)$ running from $-\varepsilon$ to $-\varepsilon + i(c + \delta)$, to $+\varepsilon + i(c + \delta)$, to ε, and back to $-\varepsilon$. Since $\exp(i\ell y)$ is analytic on and inside L we get

$$\int_{L(\varepsilon,\delta)} e^{2i\ell y}\{\tilde{T}_+(\ell)g_0(x,\ell) - 1\}d\ell = \int_L e^{2i\ell y}\tilde{T}_+(\ell)e^{i(\ell-k)x}h_-(x,k)d\ell$$

$$= \int_P e^{2i\ell(k)y}T_+(k)e^{i(\ell-k)x}h_-(x,k)\frac{k}{\ell}\,dk$$

where P is the image of L under the map $\ell \rightarrow k(\ell)$. Now P is homotopic to P' as shown below in the region $\{k : \operatorname{Im} k > 0\} \sim \{i\kappa_j\}$ where the integrand is analytic as a function of k



So

$$\int_L e^{2i\ell y}\{\tilde{T}_+(\ell)g_0(x,\ell) - 1\}d\ell = \int_{P'} e^{2i\ell y}T_+(k)e^{i(\ell-k)x}h_-(x,k)\frac{k}{\ell}\,dk.$$

Note that $kT_+/\ell = T_-$. Still exploiting the analyticity of the integrand in $0 < \operatorname{Im} k \leq \delta < \kappa_1$, we first let $\varepsilon \to 0$ and then $\delta \to 0$ in P', to get

$$\int_L e^{2i\ell y}\{\tilde{T}_+(\ell)g_0(x,\ell) - 1\}d\ell = \int_{k=-c}^{c} e^{2i\ell y}T_-(k)e^{i(\ell-k)x}h_-(x,k)dk$$

where $\ell = \ell(k) = i\sqrt{c^2 - k^2}$ since $-c < k < c$. Splitting the interval $[-c,c]$ at 0 and using (Y.1) we get

$$\int_L e^{2i\ell y}\{\tilde{T}_+g_0 - 1\}d\ell = \int_0^c e^{-2\sqrt{c^2-\sigma^2}y}e^{-\sqrt{c^2-\sigma^2}x}e^{i\sigma x}|T_-(\sigma)|^2Q(\sigma)d\sigma$$

where

$$Q(\sigma) = \frac{h_-(x,-\sigma)}{T_-(\sigma)} + \frac{e^{-2i\sigma x}h_-(x,\sigma)}{T_-^*(\sigma)}.$$

But by (Y.3) with $0 < \sigma < c$ we get $R_- = T_-/T_-^*$, whence

$$Q(\sigma) = \frac{h_-(x,-\sigma) + R_-(\sigma)e^{-2ikx}h_-(x,\sigma)}{T_-(\sigma)} = g_-(x,\sigma).$$

Appealing to the definition of h_0 again, we get

$$e^{i\sigma x}Q(\sigma) = e^{i\sigma x}g_-(x,\sigma) = h_0(x,\ell(\sigma))e^{i\ell(\sigma)x} = h_0(x,\ell(\sigma))e^{-\sqrt{c^2-\sigma^2}x}$$

whence

$$\int_L e^{2i\ell y}\{\tilde{T}_+g_0 - 1\}d\ell = \int_0^c e^{-2\sqrt{c^2-\sigma^2}(x+y)}|T_0(\sigma)|^2h_0(x,\ell(\sigma))d\sigma.$$

This finishes (K.5.2).

It remains to verify (K.5.3). For $0 \neq k \in \mathbf{R}$ the representations of g_- and R_- yield

$$T_-(k)g_-(x,k) - 1 = h_-(x,-k) - 1 + R_-(k)e^{-2ikx}h_-(x,k) = \int_{-\infty}^{\infty} \mathcal{B}_-(x,y)e^{2iky}dy$$

where

$$\mathcal{B}_-(x,y) = B_-(x,y) + F_-(x+y) + \int_{-\infty}^{\infty} B_-(x,z)F_-(x+y+z)dz.$$

Now one can check that $\mathcal{B}_-(x,y)$ is an L^2 function of y for each x. Thus

$$T_-(k)g_-(x,k) - 1 = \int_0^{\infty} \mathcal{B}_-(x,y)e^{2iky}dy + \int_{-\infty}^{0} \mathcal{B}_-(x,y)e^{2iky}dy$$

where the first term is H^{2+} in k and the second term may be computed explicitly.

The Marchenko equation (4.3) says that if $y < 0$, then

$$\begin{aligned}\mathcal{B}_-(x,y) &= -G_-(x+y) - \int_{-\infty}^{0} B_-(x,z)G_-(x+y+z)dz \\ &= -2\sum_J c_{-j}e^{+2\kappa_j(x+y)}\left\{1 + \int_{-\infty}^{0} B_-(x,z)e^{-2\kappa_j z}dz\right\} \\ &= -2\sum_J c_{-j}e^{2\kappa_j(x+y)}h_-(x,i\kappa_j).\end{aligned}$$

Thus

$$\begin{aligned}\int_{-\infty}^{0} \mathcal{B}_-(x,y)e^{2iky}dy &= -2\sum c_{-j}e^{2\kappa_j x}h_-(x,i\kappa_j)\int_{-\infty}^{0} e^{2(\kappa_j+ik)y}dy \\ &= i\sum_J c_{-j}e^{2\kappa_j x}h_-(x,i\kappa_j)/(k-i\kappa_j).\end{aligned}$$

Therefore

$$T_-(k)g_-(x,k) - 1 = \int_0^{\infty} \mathcal{B}_-(x,y)e^{2iky}dy + i\sum_J c_{-j}e^{2\kappa_j x}h_-(x,i\kappa_j)/(k-i\kappa_j). \tag{5.9}$$

So $T_-g_- - 1$ is meromorphic in Im $k > 0$. By (Y.3) it follows that $g_-(x,k)$ itself is analytic in Im $k > 0$. The relation (5.9) will be used many times in this chapter.

By applying the definitions we get

$$\begin{aligned}\operatorname{Res}(\tilde{T}_+(\ell)g_0(x,\ell),i\lambda_j) &= g_0(x,i\lambda_j)\operatorname{Res}(T_+(k(\ell)),i\lambda_j) \\ &= e^{(\kappa_j-\lambda_j)x}h_-(x,i\kappa_j)\operatorname{Res}(T_-,i\kappa_j).\end{aligned}$$

Lemma 5.4. *A sufficient condition for the part of (Y.7) asking that* Im $f_-(x,k)$ *restricted to* **R** *have a derivative at* $k = 0$ *is that*

$$\int_{-\infty}^{X} |F_-'(x)|(1+|x|^2)dx < \infty \qquad \textit{for all finite } X. \tag{5.10}$$

Proof. Since $f_-(x,k) = e^{-ikx}h_-(x,k)$ it is enough to show that $h_-(x,k)$, as a function of real k, has a derivative at $k = 0$. Now

$$h_-(x,k) = 1 + \int_{-\infty}^{0} B_-(x,y)e^{-2iky}dy.$$

So we study

$$\begin{aligned}\frac{h_-(x,k) - h_-(x,0)}{k} &= \int_{-\infty}^{0} B_-(x,y)\left\{\frac{e^{-2iky}-1}{k}\right\}dy \\ &= -\int_{-\infty}^{0} 2iyB_-(x,y)\left\{\frac{\sin 2ky}{2ky} + \frac{i\cos(2ky)-1}{2ky}\right\}dy.\end{aligned}$$

Under (5.10) we know that $yB_-(x,y)$ is integrable over $-\infty < y < 0$. Keeping $k \in \mathbf{R}$ we may apply the Dominated Convergence Theorem to get

$$\lim_{\substack{k\to 0\\ k\in R}} \frac{h_-(x,k) - h_-(x,0)}{k} = -\int_{-\infty}^{0} 2iyB_-(x,y)dy. \qquad \square$$

Note that (Y.7) imposed an added condition only on T_-, R_-, f_- and that (5.10) also treats only F_-.

Using (5.9) we get,

$$\operatorname{Res}(T_-g_-,i\kappa_j) = ic_{-j}e^{2\kappa_j x}h_-(x,i\kappa_j).$$

So

$$h_-(x,i\kappa_j) = \operatorname{Res}(T_-g_-,i\kappa_j)/ic_{-j}e^{2\kappa_j x} = g_-(x,i\kappa_j)\operatorname{Res}(T_-,i\kappa_j)/ic_{-j}e^{2\kappa_j x}.$$

Therefore

$$\operatorname{Res}(\tilde{T}_+g_0,i\lambda_j) = \frac{e^{(\kappa_j-\lambda_j)x}g_-(x,i\kappa_j)[\operatorname{Res}(T_-,i\kappa_j)]^2}{ic_{-j}e^{2\kappa_j x}}.$$

But by (Y.3)

$$[\operatorname{Res}(T_-,i\kappa_j)]^2 = -c_{+j}c_{-j}.$$

So

$$\operatorname{Res}(\tilde{T}_+g_0,i\lambda_j) = e^{-\kappa_j x}e^{-\lambda_j x}g_-(x,i\kappa_j)ic_{+j} = ic_{+j}e^{2\lambda_j x}h_0(x,i\kappa_j)$$

which completes (K.5.3). $\square$

In order to show that $h_0(x,\ell) - 1 \in H^{2+}$ as a function of ℓ, we extend the list of conditions on $T_\pm$, $R_\pm$ by adding the following assumption:

(Y.7) *Either* (A) $T_-(k)$ extends continuously to $k = 0$ so that $T_-(0) = 0$ and $T_-^{\cdot}(0) \equiv \lim_{\substack{k\to 0\\ \operatorname{Im}k\geq 0}} T_-(k)/k = i\alpha$ for $0 \neq \alpha \in \mathbf{R}$;

R_- extends continuously to $k = 0$, $R_-(0) = -1$, and R_- is differentiable at 0;

The restriction of $\operatorname{Im} f_-(x,k)$ to real k is differentiable at $k = 0$;

or (B) $T_-(k)$ extends continuously to $k = 0$ and $T_-(0) \neq 0$.

Lemma 5.5. *Suppose (Y.1-Y.7) hold. Then $g_-(x,k)$ extends into* $\operatorname{Im} k \geq 0$ *so that $g(x,k)$ is continuous in* $\operatorname{Im} k \geq 0$ *and $g_-(x,k) - 1 \in H^{2+}$ for each fixed x.*

Proof. Keep x fixed throughout. For $0 \neq k \in \mathbf{R}$, set $\Phi(x,k) = T_-(k)g_-(x,k) - 1$. Then by (5.9) we know that

$$\Phi(x,k) = \int_0^\infty \mathcal{B}_-(x,y)e^{2iky}dy + i\sum_J c_{-j}e^{2\kappa_j x}h_-(x,i\kappa_j)/(k - i\kappa_j). \tag{5.11}$$

The integral term is in H^{2+} since $\mathscr{B}_-(x,\cdot) \in L^2(\mathbf{R}^+)$. The remaining sum is clearly meromorphic in $\mathbf{C}$ with simple poles at the $i\kappa_j$, and is $O(k^{-1})$ as $|k| \to \infty$. It follows that

$$\Phi_1(x,k) \equiv \Phi(x,k) \prod_J (k - i\kappa_j)/(k + i\kappa_j) \in H^{2+}. \tag{5.12}$$

We extend $g_-(x,k)$ to $\operatorname{Im} k \geq 0$ by setting

$$g_-(x,k) = \{\Phi(x,k) + 1\}/T_-(k).$$

Since the poles of T_- cancel those of Φ and since T_- never vanishes in $\{k: \operatorname{Im} k \geq 0\} \sim \{0\}$, it follows that $g_-(x,k)$ is analytic in $\{k: \operatorname{Im} k > 0\}$.

The rest of the proof is done separately for the cases (Y.7A) and (Y.7B).

Suppose now that (Y.7B) holds. We prove first that $g_-(x,k) - 1 \in H^{2+}$ and then show continuity in $\operatorname{Im} k \geq 0$. Under (Y.7B), $T_-(k)$ is continuous in $\{k: \operatorname{Im} k \geq 0\} \sim \{i\kappa_j : j \in J\}$ and is bounded away from zero there. Write

$$g_-(x,k) - 1 = \frac{\Phi(x,k)}{T_-(k)} - \left\{1 - \frac{1}{T_-(k)}\right\}.$$

Note that $1/T_-(k)$ is bounded for $\operatorname{Im} k \geq 0$ and analytic for $\operatorname{Im} k > 0$. Since the poles cancel,

$$\frac{1}{T_-(k)} \sum_J c_{-j} e^{2\kappa_j x} h_-(x,i\kappa_j)/(k - i\kappa_j)$$

is analytic in $\operatorname{Im} k > 0$, continuous in $\operatorname{Im} k \geq 0$, and $O(k^{-1})$ at ∞. So this quantity is in H^{2+}. It is also clear that

$$\frac{1}{T_-(k)} \int_0^{+\infty} \mathscr{B}_-(x,y) e^{2iky}\, dy \in H^{2+}$$

and that therefore $\Phi(x,k)/T_-(k) \in H^{2+}$. Now $\{1 - 1/T_-\}$ is also in H^{2+} because it is analytic in $\operatorname{Im} k > 0$, continuous in $\operatorname{Im} k \geq 0$, and $O(k^{-1})$ at ∞. We conclude that $g_-(x,k) - 1 \in H^{2+}$.

It remains to show that $g_-(x,k)$ is continuous in $\operatorname{Im} k \geq 0$. Since

$$g_-(x,k) - 1 \in H^{2+}$$

it suffices to show that the restriction of $g_-(x,k) - 1$ to $\mathbf{R}$ is continuous. But by definition for $k \in \mathbf{R}$

$$g_-(x,k) = \frac{h_-(x,-k) + R_-(k)e^{-2ikx} h_-(x,k)}{T_-(k)}$$

and this is continuous on $\mathbf{R}$ under (Y.7B).

Next we consider the case that (Y.7A) holds, and look first at the continuity of g_- when restricted to the real k-axis. By definition $g_-(x,k)$ is continuous on $\mathbf{R} \sim \{0\}$. For $0 \neq k \in \mathbf{R}$, it follows from the definition that

$$g_-(x,k) = e^{-ikx}\frac{k}{T_-(k)}\left\{\frac{f_-(x,-k) - f_-(x,k)}{k} + f_-(x,k)\frac{1 + R_-(k)}{k}\right\}.$$

Applying (Y.7A) we find that $g_-(x,k)$ has a finite limit as $k \to 0$, $k \in \mathbf{R}$. It follows that both $g_-(x,k)$ and $\Phi(x,k) = g_-(x,k)T_-(k) - 1$ are continuous on $\mathbf{R}$. Therefore Φ_1 is continuous on $\mathbf{R}$. But $\Phi_1(x,k)$ is in H^{2+}, so $\Phi_1(x,k)$ is continuous in $\operatorname{Im} k \geq 0$. It follows that $\Phi(x,k)$ itself is continuous on $\{k : \operatorname{Im} k \geq 0\} \sim \{i\kappa_j : j \in J\}$, and that $g_- = \{\Phi + 1\}/T_-$ is continuous on $\{k : \operatorname{Im} k \geq 0\} \sim \{0\}$.

It remains to get continuity at $k = 0$. Since $T_-(0) = 0$,

$$\Phi(x,0) = T_-(0)g_-(x,0) - 1 = -1,$$

where $g_-(x,0)$ is the limit along the real axis, which we know is finite. To see that $\Phi(x,k)$ is differentiable at $k = 0$ as a function of real k note that

$$\frac{d}{dk}(\Phi(x,k))|_{k=0} = \lim_{\substack{k\to 0\\ k\in\mathbf{R}}}\frac{\Phi(x,k) + 1}{k} = \lim_{\substack{k\to 0\\ k\in\mathbf{R}}}\frac{T_-(k)}{k}g_-(x,k) = T_-^{\cdot}(0)g_-(x,0).$$

But now it follows that $\Phi_1(x,k)$ is differentiable at $k = 0$ as a function of real k. But $\Phi_1 \in H^{2+}$. So, by the Poisson formula.

$$\lim_{\substack{k\to 0\\ \operatorname{Im} k\geq 0}}\frac{\Phi_1(x,k) - \Phi_1(x,0)}{k} \text{ exists.}$$

By (5.12) Φ is the product of Φ_1 and a function which is analytic at $k = 0$. So

$$\lim_{\substack{k\to 0\\ \operatorname{Im} k\geq 0}}\frac{\Phi(x,k) + 1}{k} = \lim_{\substack{k\to 0\\ \operatorname{Im} k\geq 0}}\frac{\Phi(x,k) - \Phi(x,0)}{k} \text{ exists,} \tag{5.13}$$

whence

$$\lim_{\substack{k\to 0\\ \operatorname{Im} k\geq 0}} g_-(x,k) = \lim_{\substack{k\to 0\\ \operatorname{Im} k\geq 0}}\frac{k}{T_-(k)}\frac{\Phi(x,k) + 1}{k} \text{ exists,}$$

and $g_-(x,k)$ is continuous in $\operatorname{Im} k \geq 0$.

To show that $g_-(x,k) - 1 \in H^{2+}$, note that

$$\begin{aligned} g_-(x,k) - 1 &= \frac{\Phi(x,k) + 1 - T_-(k)}{T_-(k)} \\ &= \left(\prod_J \frac{k - i\kappa_j}{k + \kappa_j}\right)\frac{(k + i)}{1}\left(\frac{\Phi(x,k) + 1 - T_-(k)}{k}\right) \\ &\quad\times\left\{\frac{k}{T_-(k)}\cdot\frac{1}{k + i}\cdot\left(\prod_J \frac{k - i\kappa_j}{k + \kappa_j}\right)\right\}. \end{aligned}$$

Clearly the factor in curly brackets, $\{\ \ldots\ \}$, is continuous and bounded on

$\operatorname{Im} k \geq 0$ and is analytic in $\operatorname{Im} k > 0$. Therefore to get $g_-(x,k) - 1 \in H^{2+}$ it is sufficient to show that

$$\Psi(x,k) \equiv \left(\prod_j \frac{k - i\kappa_j}{k + \kappa_j}\right) \frac{(k + i)}{1} \frac{\Phi(x,k) + 1 - T_-(k)}{k} \in H^{2+}. \tag{5.14}$$

Since $T_-(k)/k$ and $\{\Phi(x,k) + 1\}/k$ have finite limits as $k \to 0$, $\operatorname{Im} k \geq 0$ by (5.14) and (Y.7A), $\Psi(x,k)$ is continuous in $\operatorname{Im} k \geq 0$. Clearly Ψ is analytic in $\operatorname{Im} k > 0$. To show that $\Psi \in H^{2+}$ it remains to show

$$\sup_{\eta > 0} \int_{\xi = -\infty}^{+\infty} |\Psi(x, \xi + i\eta)|^2 d\xi < \infty.$$

Define $\chi(k)$ on $\operatorname{Im} k \geq 0$ by

$$\chi(k) = \begin{cases} 1 & \text{if } \operatorname{Im} k > 1 \text{ or if } |\operatorname{Re} k| > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly

$$\sup_{\eta > 0} \int_{-\infty}^{\infty} |\{1 - \chi(\xi + i\eta)\} \Psi(x, \xi + i\eta)|^2 d\xi < \infty$$

since $1 - \chi$ has compact support. To see why

$$\sup_{\eta > 0} \int_{-\infty}^{\infty} \chi(\xi + i\eta) |\Psi(\xi + i\eta)|^2 d\xi < \infty$$

we use the decomposition

$$\Psi(k) = \left(\frac{k + i}{k}\right) \Phi_1(x,k) + \left(\frac{k + i}{k}\right)\left(\prod_j \frac{k - i\kappa_j}{k + i\kappa_j}\right)(1 - T_-(k)).$$

Note that both

$$\Phi_1(x,k) \quad \text{and} \quad \left(\prod \frac{k - i\kappa_j}{k + i\kappa_j}\right)(1 - T_-(k))$$

are in H^{2+} and that $(k + i)/k$ is bounded on the support of χ. It now follows that $\Psi \in H^{2+}$ and that $g_-(x,k) - 1 \in H^{2+}$. □

In order to prepare for the next theorem we describe a result of Carleson [9, p. 33]. A "Carleson measure" m is a positive measure supported in the upper half-plane such that

$$\sup_{h > 0} \left[\sup \left\{ \frac{m(Q)}{h} : Q \in S(h) \right\} \right] \equiv N(m) < \infty$$

where $S(h)$ is the family of squares of the form

$$Q = \{\alpha + i\beta : \alpha_0 < \alpha < \alpha_0 + h, 0 < \beta < h\}.$$

The supremum $N(m)$ is called the "Carleson norm" of m. We use the following result of Carleson: There is a constant C such that for all $g \in H^{2+}$ and for all Carleson measures m

$$\int |g|^2 dm \leq C\, N(m) \|g\|^2 \tag{5.15}$$

where $\|g\|$ is the H^{2+} norm of g.

Suppose $g = g(k)$ is in H^{2+} as a function of k and let $h(\ell) = g(k(\ell))$. We need to analyze the question whether the integrals of $|h|^2$ over the lines $\operatorname{Im} \ell = \mu$ are uniformly bounded for $0 < \mu < \infty$. The image of $\{\ell : \operatorname{Im} \ell = \mu\}$ under the map $\ell \to k$ is the graph of the function $\tilde{\beta}$, defined by

$$\beta = \tilde{\beta}(\alpha) \equiv \mu \sqrt{\frac{\alpha^2 + \mu^2 - c^2}{\alpha^2 + \mu^2}}. \tag{5.16}$$

The curve will be called γ_μ. Note that if $\mu \geq c$, then $\tilde{\beta}$ is defined for all α but that if $0 < \mu < c$, then $\tilde{\beta}$ is defined only for $|\alpha| \geq \sqrt{c^2 - \mu^2} \equiv \alpha_0$. Now for fixed μ, the map $\lambda + i\mu \to \alpha + i\tilde{\beta}(\alpha)$ is one-to-one. The inverse is given by

$$\lambda = \tilde{\lambda}(\alpha) \equiv \alpha \sqrt{\frac{\alpha^2 + \mu^2 - c^2}{\alpha^2 + \mu^2}}. \tag{5.17}$$

Thus the change of variable formula gives

$$\int_{-\infty}^{\infty} |h(\lambda + i\mu)|^2 d\lambda = \int_{\operatorname{Dom} \tilde{\beta}} |g(\alpha + i\tilde{\beta}(\alpha))|^2 \left|\frac{d\tilde{\lambda}}{d\alpha}\right| d\alpha. \tag{5.18}$$

Computation shows that

$$\frac{d\tilde{\lambda}}{d\alpha} = \sqrt{\frac{\alpha^2 + \mu^2 - c^2}{\alpha^2 + \mu^2}} \left\{\frac{\mu^2}{\alpha^2 + \mu^2}\right\} + \frac{|\alpha|}{\sqrt{\alpha^2 + \mu^2}} \frac{|\alpha|}{\sqrt{\alpha^2 + \mu^2 - c^2}}.$$

Thus $d\tilde{\lambda}/d\alpha$ is defined and positive for $|\alpha| > \alpha_0$. We can interpret the right side of (5.18) as

$$\int |g|^2 d\tilde{m}_\mu$$

where $\tilde{m}_\mu$ is the measure supported on γ_μ with density $|d\tilde{\lambda}/d\alpha| = d\tilde{\lambda}/d\alpha$. If all $\tilde{m}_\mu$ were Carleson measures with Carleson norms $N(\tilde{m}_\mu)$ uniformly bounded above by A, then we would get

$$\int_{-\infty}^{\infty} |h(\lambda + i\mu)|^2 d\lambda = \int |g|^2 d\tilde{m}_\mu \leq CA\|g\|^2.$$

Unfortunately, while the $\tilde{m}_\mu$ are all Carleson measures, the norms $N(\tilde{m}_\mu)$ blow up like μ^{-1} as $\mu \to 0$. Thus in the theorem below we must argue more carefully.

Theorem 5.6. *Under the hypotheses (Y.1-Y.7), the function* $h_0(x,\ell) - 1$ *belongs to* H^{2+} *as a function of* ℓ.

Proof. Recall that by definition

$$h_0(x,\ell) = e^{-i\ell x} e^{ikx} g_-(x,k(\ell))$$

for $\operatorname{Im} \ell \geq 0$ with $\ell \notin \{i\lambda : 0 \leq \lambda \leq c\}$. Set

$$h(x,\ell) = h_0(x,\ell) - 1$$

and

$$g(x,k) = h(x,k(\ell)) = e^{-i\ell(k)x} e^{ikx} g_-(x,k) - 1.$$

By Lemma 5.5 we know $g_-(x,k) - 1$ is in H^{2+} as a function of k. From the decomposition

$$g(x,k) = e^{-i\ell x} e^{ikx}[g_-(x,k) - 1] + [e^{-i\ell x} e^{ikx} - 1]$$

it follows that $g(x,k)$ is in H^{2+}, too. Clearly $h(x,\ell) = g(x,k)\ell))$.

To show that $h(x,\ell)$ is in H^{2+}, we first show that h is well defined and continuous in $\operatorname{Im} \ell \geq 0$ and analytic in $\operatorname{Im} \ell > 0$. It is sufficient to show that

$$e^{ikx} g_-(x,k) = e^{-ikx} g_-(x,-k) \tag{5.19}$$

for $k \in \mathbf{R}$ with $0 \neq |k| \leq c$. But for these k

$$e^{ikx} g_-(x,k) = \frac{e^{ikx} h_-(x,-k)}{T_-(k)} + \frac{e^{-ikx} h_-(x,k)}{T_-^*(k)} = 2 \operatorname{Re}[e^{ikx} h_-(x,-k)/T_-(k)].$$

By (Y.1), relation (5.19) now follows and $h(x,\ell)$ is continuous on $\{i\lambda : 0 \leq \lambda \leq c\}$. But since $e^{ikx} g_-(x,k)$ is real valued for $0 < k \leq c$ and is continuous at $k = 0$, it must be real there. Thus $h(x,\ell)$ is continuous at $\ell = ic$, too.

In order to conclude that $h \in H^{2+}$ it remains to verify that

$$\sup_{\mu>0} \left\{ \int_{-\infty}^{\infty} |h(x,\lambda + i\mu)|^2 d\lambda \right\} < \infty.$$

We introduce the following measures m_μ: if $c \leq \mu < \infty$, then m_μ is supported on the curve γ_μ and has density $|d\tilde{\lambda}/d\alpha|$; if $0 < \mu < c$, then m_μ is supported on the part of γ_μ where $|\alpha| \geq 2c$ and has density $|d\tilde{\lambda}/d\alpha|$. One can easily verify that these are Carleson measures with Carleson norms $N(m_\mu)$ uniformly bounded by 3 for $0 < \mu < \infty$. We now exploit Carleson's result (5.15) and the formula (5.18). If $c \leq \mu < \infty$, then

$$\int_{-\infty}^{\infty} |h(x,\lambda + i\mu)|^2 d\lambda \leq \int |g|^2 dm_\mu \leq C3\|g\|^2.$$

If $0 < \mu < c$, then we split the integral as follows:

$$\int_{-\infty}^{\infty} |h(x,\lambda + i\mu)|^2 d\lambda = \int_{\operatorname{Dom} \tilde{\beta}} |g(x,\alpha + i\tilde{\beta}(\alpha))|^2 |d\tilde{\lambda}/d\alpha| d\alpha = I_1 + I_2$$

where

$$I_1 = \int_{|\alpha|>2c}^{\infty} |g(x,\alpha + i\tilde{\beta}(\alpha))|^2 |d\tilde{\lambda}/d\alpha| d\alpha = \int |g|^2 dm_\mu \leq 3C\|g\|^2$$

and

$$I_2 = \int_{\sqrt{c^2-\mu^2}<|\alpha|<2c} |g(x,\alpha + i\tilde{\beta}(\alpha))| \, |d\tilde{\lambda}/d\alpha| d\alpha.$$

When $0 < \mu < c$, the curve $\beta = \tilde{\beta}(\alpha)$ satisfies $|\tilde{\beta}(\alpha)| \leq \mu < c$ for $|\alpha| \leq 2c$. Now $g(x,k)$ is continuous in Im $k \geq 0$, so there is a constant M^2 such that

$$\sup\{|g(x,\alpha + i\beta)|^2 : |\alpha| \leq 2c, 0 \leq \beta \leq c\} = M^2 < \infty.$$

Thus, keeping $0 < \mu < c$,

$$I_2 \leq \int_{\sqrt{c^2-\mu^2}<|\alpha|<2c} M^2 |d\tilde{\lambda}/d\alpha| d\alpha.$$

But $d\tilde{\lambda}/d\alpha$ is positive, so we can integrate explicitly to get

$$\begin{aligned} I_2 &\leq M^2\{\tilde{\lambda}(2c) - \tilde{\lambda}(\sqrt{c^2 - \mu^2}) + \tilde{\lambda}(-\sqrt{c^2 - \mu^2}) - \tilde{\lambda}(-2c)\} \\ &= M^2\{\tilde{\lambda}(2c) - \tilde{\lambda}(-2c)\} = 2M^2\tilde{\lambda}(2c) \leq 2M^2 2c. \end{aligned}$$

Thus

$$I_1 + I_2 \leq 3C\|g\|^2 + 4M^2 c \qquad \text{for } 0 < \mu < c.$$

It follows that

$$\sup_{\mu>0} \int |h(x,\lambda + i\mu)|^2 d\lambda < \infty$$

and that $h(x,\ell) \in H^{2+}$ as a function of ℓ. □

Theorem 5.7. *Suppose that the candidate scattering data satisfy (Y.1-Y.7) and that v_+ and v_- are defined by (4.5, 4.6). Then $v_+(x) = v_-(x)$.*

Proof. It is straightforward to check that if (Y.1-Y.7) hold, then (X.1-X.6) hold, too. So we apply Theorems 5.2, 5.3, and 5.6 to conclude that $f_+(x,k)$ in a linear combination of $f_\pm^*(x,k)$ and $f_-(x,k)$ for $k \in \mathbf{R}$, $|k| > c$. By Lemma 5.1, it follows that $v_+ = v_-$. □

Section 6. Conclusion of inverse scattering argument. Assume that (Y.1-Y.7) hold. Consider

$$v(x) = v_+(x) = v_-(x)$$

as defined in Section 4. By the analysis of the Marchenko equation [1], we know that

$$\int_{-\infty}^{X} |v_-(x)|(1 + |x|)dx = \int_{-\infty}^{X} |\partial_x B_-(x,0)|(1 + |x|)dx < \infty$$

for all finite X and that

$$\int_{X}^{\infty} |v_+(x) - c^2|(1 + |x|)dx = \int_{X}^{\infty} |\partial_x B_+(x,0)|(1 + |x|)dx < \infty$$

for all finite X. It follows that

$$\int_{-\infty}^{\infty} |v(x) - c^2 H(x)|(1 + |x|)dx < \infty.$$

Therefore $v(x)$ belongs to the class of potentials $P(x,1)$.

Theorem 6.1. *Suppose that the candidate data (4.1) satisfy conditions (Y.1-Y.7) and that the function v is constructed from the scattering data as in Section 4. Then v is a potential of class* $P(c,1)$ *and the scattering data of v are exactly the objects given in (4.1).*

Proof. We have already seen that $v \in P(c,1)$. The functions $f_\pm(x,k)$ are the Jost functions of v since they solve

$$-y'' + v(x)y = k^2 y$$

and have the correct asymptotic behavior

$$f_+(x,k) \sim e^{i\ell x} \quad \text{at } +\infty; \qquad f_-(x,k) \sim e^{ikx} \quad \text{at } -\infty.$$

By (4.8±) we see that T_+, T_-, R_+, R_- are the transmission and reflection coefficients of v.

The numbers $i\kappa_j$ are the poles of T_- by assumption (Y.3).

Next we show that the μ_j satisfy the relation $f_-(x,i\kappa_j) = \mu_j f_+(x,i\kappa_j)$ for $j \in J$.

Fix a j in J and take the residue at $i\kappa_j$ of both sides of (5.9) to obtain

$$g_-(x,i\kappa_j)\text{Res}(T_-,i\kappa_j) = ic_{-j}e^{2i\kappa_j}h_-(x,i\kappa_j). \tag{6.1}$$

Since $h_+ = h_0 = e^{ikx}e^{-i\ell x}g_-$, we get

$$g_-(x,i\kappa_j) = e^{i\ell(i\kappa_j)x}e^{-i(i\kappa_j)x}h_+(x,i\kappa_j) = e^{\kappa_j x}f_+(x,i\kappa_j).$$

By (Y.3) and the definition $c_{-j} = c_{+j}/\mu_j^2$,

$$\text{Res}(T_-,i\kappa_j) = ic_{+j}/\mu_j.$$

Also note that

$$e^{\kappa_j x}h_-(x,i\kappa_j) = e^{-i(i\kappa_j)x}h_-(x,i\kappa_j) = f_-(x,i\kappa_j).$$

Therefore (6.1) yields

$$e^{\kappa_j x}f_+(x,i\kappa_j)ic_{+j}/\mu_j = ic_{+j}e^{\kappa_j x}f_-(x,i\kappa_j)/\mu_j^2$$

whence

$$f_-(x,i\kappa_j) = \mu_j f_+(x,i\kappa_j).$$

Finally we need to verify that the given c_{+j} are the norming constants of the $f_+(x,i\kappa_j)$. From the forward scattering analysis of v we obtain

$$\mathrm{Res}(T_-,i\kappa_j) = i/\mu_j \int_{-\infty}^{\infty} |f_+(x,i\kappa_j)|^2 dx.$$

However, by (Y.3)

$$\mathrm{Res}(T_-,i\kappa_j) = ic_{+j}/\mu_j$$

whence

$$c_{+j} = 1\Big/\int_{-\infty}^{\infty} |f_+(x,i\kappa_j)|^2 dx. \qquad \square$$

Note that in the inverse problem the disjunction between (Y.7A) and (Y.7B) corresponds to the division between generic and exceptional cases: In case (Y.7A), the Wronskian $W[f_-,f_+] \neq 0$ at $k = 0$ since

$$2ik = T_-(k)W[f_-,f_+] \tag{6.2}$$

yields

$$2i = T_-^{\cdot}(0)W(0) + T_-(0)W^{\cdot}(0) = i\alpha W(0).$$

In case (Y.7B), we evaluate (6.2) at $k = 0$ to get $W[f_-,f_+] = 0$ when $k = 0$.

We conclude this section by clarifying the extent to which we have succeeded in characterizing the scattering data associated with potentials of class $P(c,N)$. For $N \geq 2$ we have a characterization of the scattering data for potentials in $P(c,N)$. For $N = 1$, the sufficient conditions (Y.1-Y.7) are stronger than the necessary conditions (C.1-C.6) of Theorem 2.1.

Theorem 6.2. *The following are necessary and sufficient conditions that the objects (4.1) are the scattering data of a potential in $P(c,N)$ with $N \geq 2$:*

(Z.1) *$T_\pm(-k) = T_\pm^*(k)$ and $R_\pm(-k) = R_\pm^*(k)$ for $0 \neq k \in \mathbf{R}$.*

(Z.2) *$R_-(k)$ is continuous on $\mathbf{R}$; $R_+(k)$ is continuous on $\mathbf{R} \sim (-c,c)$. $\tilde{R}_+(\ell) \in L^2(\mathbf{R})$.*
$T_\pm(k)$ are meromorphic on $\{k : \mathrm{Im}\, k > 0\}$ and continuous on $\{k : 0 \leq \mathrm{Im}\, k < \underline{\kappa}\} \sim \{0\}$ where $\underline{\kappa} = \inf \kappa_j$.

(Z.3) *The poles of T_+ and of T_- are the same, namely the $i\kappa_j$. These poles are all simple and*

$$\mathrm{Res}(T_-,i\kappa_j) = c_{+j}/\mu_j.$$

(Z.4) *(i) If* $\operatorname{Im} k \geq 0$, $k \neq 0$, *and* $k \notin \{i\kappa_j : j \in J\}$, *then* $kT_+(k) = \ell T_-(k)$.
(ii) If $k \in \mathbf{R}$ *and* $|k| > c$, *then*

$$1 = \frac{k}{\ell}|T_+|^2 + |R_+|^2 = \frac{\ell}{k}|T_-|^2 + |R_-|^2 \qquad 0 = \ell T_- R_-^* + kT_+^* R_+.$$

(iii) If $k \in \mathbf{R}$ *and* $0 < |k| \leq c$, *then* $R_-(k) = T_-(k)/T_-^*(k)$.
(iv) If $\operatorname{Im} k \geq 0$, $k \neq 0$, *and* $k \notin \{i\kappa_j : j \in J\}$, *then* $T_-(k) \neq 0$.

(Z.5) $T_\pm(k) = 1 + O(k^{-1})$ *as* $|k| \to \infty$ *in* $\operatorname{Im} k \geq 0$,
$R_\pm(k) = O(k^{-1})$ *as* $|k| \to \pm\infty$, $k \in \mathbf{R}$.

(Z.6) *If* F_+ *and* F_- *are the* L^2 *functions*

$$F_-(x) = \pi^{-1}\int_{-\infty}^{\infty} R_-(k)e^{-2ikx}\,dk$$

$$F_+(x) = \pi^{-1}\int_{-\infty}^{\infty} \tilde{R}_+(\ell)e^{2i\ell x}\,d\ell$$

then F_+ *and* F_- *are absolutely continuous, and*

$$\int_{-\infty}^{X} |F_-'(x)|(1 + |x|^N)\,dx < \infty$$

and

$$\int_{X}^{\infty} |F_+'(x) + H_+'(x)|(1 + |x|^N)\,dx < \infty$$

for all finite X.

(Z.7) *Either (A)* $T_-(k)$ *extends continuously to* $k = 0$, $T_-(0) = 0$, *and* $T_-^{\cdot}(0) = \lim_{\substack{k\to 0\\ \operatorname{Im} k \geq 0}} T_-(k)/k = i\alpha$ *with* $0 \neq \alpha \in \mathbf{R}$;
$R_-(k)$ *extends continuously to* $k = 0$, $R_-(0) = -1$, *and* $R_-(k)$ *is differentiable at* $k = 0$;
or (B) $T_-(k)$ *extends continuously to* $k = 0$ *and* $T_-(0) \neq 0$.

Proof. The necessity of these conditions was proved in Section 2. Their sufficiency follows from Theorem 6.1 and the remark that (Z.6) implies that v satisfies (1.2).

Theorem 6.3. *Suppose* $v \in P(c,1)$; *then (C.1-C.6) hold. Conversely, if the stronger conditions (Y.1-Y.7) hold, then the candidate data are the scattering data of a potential in* $P(c,1)$.

Proof. Theorem 2.1 and Theorem 6.1.

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